MATH 424 Practice Exam I

1. For $E^n$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, let
   \[
   d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \quad d(x, y) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}, \quad d_\infty(x, y) = \max_i |x_i - y_i|.
   \]
   Find $z \in E^n$ so that $d_2(z, 0) = \sqrt{n} d_\infty(z, 0)$ and $d_1(z, 0) = \sqrt{n} d_2(z, 0)$.

2. Using the $(\epsilon, N)$ definition of limit, prove that
   \[
   \lim_{n \to \infty} \sqrt{9 - \frac{1}{n^2}} = 3.
   \]

3. (a) Suppose $(E, d)$ is a metric space and $K_1$ and $K_2$ are both compact. Prove from the definition of compactness that $K_1 \cup K_2$ is compact.
   (b) Find an infinite family of compact subsets $K_n \in \mathbb{R}$ with the property that the union of $K = \bigcup_{n=1}^{\infty} K_n$ is not compact.

4. There will be T/F questions, which you are required to justify your answer, and you will be asked to complete sentences.