In the axiomatic approach to introduce vector algebra, no attempt is made to describe the nature of a vector or the algebraic operations on vectors. Instead, vectors and vector operations are thought of as undefined concepts of which we know nothing except that they satisfy a certain set of axioms. Such an algebraic system, with appropriate axioms, is called a linear space. Examples of linear spaces occur in all branches of mathematics, and we study some of them here.

The definition of a linear space. Let $V$ denote a nonempty set of objects, called “elements”. The set $V$ is called a linear space if it satisfies the following ten axioms.

**AXIOM 1** (closure under addition) For every pair of elements $x$ and $y$ in $V$, there corresponds a unique element in $V$, called the sum of $x$ and $y$, denoted by $x + y$.

**AXIOM 2** (closure under multiplication by real number) For every $x$ in $V$ and any real number $a$, there corresponds an element in $V$, called the product of $a$ and $x$, denoted by $ax$.

**AXIOM 3** (commutative law for addition) For all $x$ and $y$ in $V$, we have $x + y = y + x$.

**AXIOM 4** (associative law for addition) For all $x$, $y$ and $z$ in $V$, we have $(x + y) + z = x + (y + z)$.

**AXIOM 5** (existence of zero element) There is an element in $V$, denoted by $0$, such that $x + 0 = x$ for all $x$ in $V$.

**AXIOM 6** (existence of negatives) For every $x$ in $V$, the element $(-1)x$ has the property $x + (-1)x = 0$.

**AXIOM 7** (associative law for multiplication) For every $x$ in $V$ and all real numbers $a$ and $b$, we have $a(bx) = (ab)x$.

**AXIOM 8** (distributive law for addition in $V$) For all $x$ and $y$ in $V$ and all real $a$, we have $a(x + y) = ax + ay$.

**AXIOM 9** (distributive law for addition of numbers) For all $x$ in $V$ and all real $a$ and $b$, we have $(a + b)x = ax + bx$.

**AXIOM 10** (existence of identity) For every $x$ in $V$, we have $1x = x$.

**Examples of linear spaces.** The following are examples of linear spaces.

**Example 1.** The set of real numbers $\mathbb{R}$ is a linear space with the ordinary addition and multiplication of real numbers.

**Example 2.** The $n$-space $\mathbb{R}^n$ is a linear space with vector addition and multiplication by scalars defined in the usual way in terms of components.

**Example 3** (Function spaces). The set of all functions $V$ defined on a given interval, say $0 \leq x \leq 1$, is a linear space with addition of two functions $f$ and $g$ defined in the usual way as $(f + g)(x) = f(x) + g(x)$ for all $0 \leq x \leq 1$, and multiplication of $f$ by a real scalar $a$ defined as $(af)(x) = af(x)$ for all $0 \leq x \leq 1$.

Each of the following sets of functions defined on $0 \leq x \leq 1$ is a linear space with addition and multiplication by scalars inherited from $V$. 

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**1.**
(a) The set of all polynomials.
(b) The set of all polynomials of degree \( \leq n \), where \( n \) is fixed.
(c) The set of all functions continuous on \( 0 \leq x \leq 1 \), denoted by \( C([0, 1]) \).
(d) The set of all functions differentiable at a (given) point, say at 0.
(e) The set of all functions integrable on \( 0 \leq x \leq 1 \).

Other examples are given in [Apo, Section 15.3], but the linear space we need most is \( V_n \).

**Exercise.** Verify that each of the sets in Example 1-3 is a linear space.

**Subspaces.** Given a linear space \( V \), a nonempty subset \( W \) of \( V \) is called a **subspace** of \( V \) if \( W \) is a linear space in its own right with the same operations of addition and multiplication by scalars as those of \( V \).

It is straightforward to see [Apo, Theorem 15.4] that a nonempty \( W \subset V \) is a subspace of \( V \) if and only if it is closed under addition and scalar multiplication. In particular, each of the spaces in Example 3 is a subspace of the function space \( V \). The most extreme possibility for a subspace is to contain only one vector, the zero vector. This is the smallest possible vector space: the empty set is not allowed. At the other extreme, the largest subspace, is the whole of the original space.

In what follows, we will work on \( V_n \), the \( n \)-space.*

Let \( S = \{ A_1, A_2, \ldots, A_k \} \) be a subset of \( V_n \). A vector of the form

\[
\sum_{i=1}^{k} c_i A_i = c_1 A_1 + c_2 A_2 + \cdots + c_k A_k,
\]

where all \( c_i \)'s are scalars, is called a **linear combination** of elements of \( S \). Linear combinations of elements of \( S \) form a subspace of \( V_n \). We call this the **linear span** of \( S \) and denote it by \( L(S) \).

**Example 4** (The unit coordinate vectors, Section 12.10 of [Apo]). In \( V_n \), the \( n \) vectors \( E_1 = (1, 0, \ldots, 0) \), \( E_2 = (0, 1, 0, \ldots, 0) \), \( \ldots, E_n = (0, 0, \ldots, 1) \) are called the **unit coordinate vectors**. Note that \( \{ E_1, E_2, \ldots, E_n \} \) spans \( V_n \). Indeed, every vector \( A = (a_1, a_2, \ldots, a_n) \) in \( V_n \) can be written in the form

\[
A = a_1 E_1 + a_2 E_2 + \cdots + a_n E_n.
\]

In \( V_2 \), the unit coordinate vectors \( E_1 \) and \( E_2 \) are often denoted, respectively, by the symbols \( i \) and \( j \). In \( V_3 \), the symbols \( i, j \) and \( k \) are also used in place of \( E_1, E_2, E_3 \).

Different sets may span the same space. For instance, \( V_2 \) is spanned by \( \{ i, j \} \), or \( \{ i, j, i+j \} \), or \( \{ 0, i, -i, j, -j \} \). A natural question then is: what is the smallest number of elements to span \( V_n \) or its subspaces? To discuss this and related questions, we introduce the concepts of **linear independence**.

**Linear independence in \( V_n \).** A set \( S = \{ A_1, \ldots, A_k \} \) in \( V_n \) is said to be **linearly independent** if

\[
\sum_{i=1}^{k} c_i A_i = 0 \quad \text{implies} \quad c_i = 0 \quad \text{for all } i.
\]

Otherwise, the set is said to be **linearly dependent**.

**Theorem 5.** \( S = \{ A_1, \ldots, A_k \} \) is linearly independent if and only if it spans every vector in \( L(S) \) uniquely.

*Extension to general linear spaces is done in [Apo, Chapter 1].
Proof: If $S$ spans every vector in $L(S)$ uniquely then it clearly spans 0 uniquely, implying that $S$ is linearly independent. Conversely, suppose $S$ is linearly independent. If

$$X = \sum_{i=1}^{k} c_i A_i = \sum_{i=1}^{k} d_i A_i,$$

then $0 = X - X = \sum_{i=1}^{k} (c_i - d_i)A_i$. Linear independence, on the other hand, implies $c_i - d_i = 0$ for all $i$, and hence $c_i = d_i$ for all $i$.

**Example 6.** (a) The set $\{E_1, \ldots, E_n\}$ in $V_n$ is linearly independent. (b) Any set containing 0 is linearly dependent. (c) A set of single element $\{A\}$ is linearly independent if and only if $A$ is nonzero. (d) $S = \{i, j, i + j\}$ in $V_2$ is linearly dependent, where $i, j$ are the unit vectors of $V_2$. More generally, any three vectors in $V_2$ are linearly dependent. (why?)

**Theorem 7.** Let $S = \{A_1, \ldots, A_k\} \subset V_n$ be linearly independent and let $L(S)$ be the linear span of $S$. Then, any $k + 1$ vectors in $L(S)$ are linearly dependent.

Proof. The proof is by Induction on $k$, the number of vectors in $S$. First, let $k = 1$ and $S = \{A_1\}$. Clearly, $A_1 \neq 0$ since $S$ is independent. Now consider two vectors $B_1$ and $B_2$ in $L(S)$. Let $B_1 = c_1 A_1$ and $B_2 = c_2 A_1$ for some scalars $c_1 \neq c_2$. It is noted that $c_2 B_1 - c_1 B_2 = 0$. This is a nontrivial representation of 0, and thus $B_1$ and $B_2$ are linearly dependent, proving the assertion.

Now we assume that the theorem is true for $k - 1$ and prove that it is also true for $k$. Let $S = \{A_1, \ldots, A_k\}$ and let us take $k + 1$ vectors in $L(S)$, say $T = \{B_1, \ldots, B_{k+1}\}$. We must prove that $T$ is linearly dependent. Since since all $B_i$ are in $L(S)$, we have

$$B_i = \sum_{j=1}^{k} a_{ij} A_j \quad \text{for each } i = 1, 2, \ldots, k + 1. \quad (1)$$

We now split the proof into two cases as to whether all scalars $a_{i1}$ that multiply $A_1$ are zero or not.

CASE 1: $a_{i1} = 0$ for every $i = 1, 2, \ldots, k + 1$. In this case, the sum in (1) does not involve $A_1$, and hence each $B_i$ in $T$ is in the linear span of the set $S_1 = \{A_2, A_3, \ldots, A_k\}$. On the other hand, $S_1$ is linearly independent and consists of $k - 1$ vectors. By the induction hypothesis, the theorem is true for $k - 1$ and so the set $T$ is dependent and we are done.

CASE 2: Now let us assume that not all the scalars $a_{i1}$ are equal to 0. By renumbering, let us assume that $a_{11} \neq 0$. Now, take $i = 1$ in (1) and multiply both sides $c_i$, where $c_i = a_{i1}/a_{11}$ to obtain

$$c_i B_1 = a_{i1} A_1 + \sum_{j=2}^{k} c_i a_{ij} A_j.$$

From this, we subtract (1) to obtain

$$c_i B_1 - B_i = \sum_{j=2}^{k} (c_i a_{1j} - a_{ij}) A_j \quad \text{for } i = 2, \ldots, k + 1.$$

This equation expresses each of the $k$ vectors $c_i B_1 - B_i$ as a linear combination of $k - 1$ linearly independent vectors $A_2, \ldots, A_k$. By the induction hypothesis, then the $k$ vectors $c_i \cdot B_1 - B_i$ must be linearly dependent. Hence, for some choice of scalars $t_2, \ldots, t_{k+1}$, not all zero, we have

$$\sum_{i=2}^{k+1} t_i c_i B_1 - B_i = 0,$$
from which we find that \[
\left( \sum_{i=2}^{k+1} t_i c_i \right) B_1 - \sum_{i=2}^{k+1} t_i B_i = 0.
\]
But, this is a nontrivial linear combination of \(B_1, \ldots, B_{k+1}\) which represents the zero vector. Therefore, the vectors \(B_1, \ldots, B_{k+1}\) must be dependent, completing the proof. \(\square\)

This motivates the definition of bases.

**Definition 8.** A finite set in \(V_n\) is called a basis for \(V_n\) if it is linearly independent and spans \(V_n\).

For example, the set of unit coordinate vectors \(\{E_1, \ldots, E_n\}\) is a basis for \(V_n\).

**Theorem 9.** (a) A basis for \(V_n\) contains exactly \(n\) vectors.
(b) Any set of \(n\) linearly independent vectors is a basis for \(V_n\).

For proof, please read the proof of Theorem 12.10 in [Apo]. The above theorem states that any two bases for \(V_n\) contains the same number of vectors, \(n\). This number, shared by all bases and expresses the number of “degree of freedom” of the space, is called the dimension of \(V_n\).

**Exercise.** Prove that any set of \(k\) vectors in \(V_n\) is linearly dependent if \(k > n\).

**References**


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