Consider the function $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$. Let us write it in the form

$$F(x, y) = F(x_1, \ldots, x_n, y_1, \ldots, y_m).$$

Consider the equation

$$F(x, y) = c,$$

where $c$ is a vector in $\mathbb{R}^n$. This equation represents a system of $n$ equations in $n+m$ unknowns. In practice, such an equation arises when a system has $n$ equations in $n$ unknowns and $m$ parameters. In general, under some reasonable assumption, one expects to be able to solve the system for $n$ of the unknowns, say $x$, in terms of the others, say $y$. That is, one expects to obtain $x = f(y)$ for some function $f$.

Once we know it is possible to solve for $x$ in terms of $y$, one can then calculate the derivative of the resulting function $f$ by using the chain rule. Apostol [Apo] explains how this is done by working through a number of examples in Section 9.6 and Section 9.7.

A natural question, though, is under which assumptions it is possible to determine $x$ as a function of $y$? In these notes, we discuss that question. Our goal is to obtain the Implicit Function Theorem. We first discuss some examples in [Apo] to motivate the statement of the theorem.

**Two examples.** Let us first discuss the example discussed on [Apo, pp. 294–295] to consider the equation of the form

$$F(x, y, z) = 0,$$

where $F$ is of class $C^1$. Assuming that one can solve this equation for $z$ as a function of $x$ and $y$, that is $z = f(x, y)$ for some function $f$, by an application of the chain rule, we obtain the partial derivatives of $f$ as

$$\frac{\partial f}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}.$$

For $F(x, y, f(x, y)) = 0$ leads to

$$\frac{\partial F}{\partial x} = D_1F \frac{\partial x}{\partial x} + D_2F \frac{\partial y}{\partial x} + D_3F \frac{\partial f}{\partial x}.$$

Similarly,

$$\frac{\partial f}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}.$$

It is readily seen that a necessary condition to carry out these calculation is $\partial F/\partial z \neq 0$. It is rather surprising that the condition $\partial F/\partial z \neq 0$ is also sufficient to justify the assumptions we made in carrying them out. This is a consequence of a celebrated theorem of analysis called the Implicit Function Theorem. More precisely, the theorem under the given setup states as follows: if $F(x_0, y_0, z_0) = 0$ and $\partial F/\partial z(x_0, y_0, z_0) \neq 0$ then there exists a unique differentiable function $f : B \to \mathbb{R}$, where $B$ is an open ball in $\mathbb{R}^2$ containing $(x_0, y_0)$, such that $f(x_0, y_0) = z_0$ and $F(x, y, f(x, y)) = 0$ for all $(x, y) \in B$.

**Example 1.** Let us consider the equation $F(x, y, z) = 0$, where

$$F(x, y, z) = x^2 + y^2 + z^2 - 4.$$

It represents the surface of the sphere with center at the origin and radius 2.
At a point on the \((x, y)\)-plane which satisfies the equation, say at \((0, 2, 0)\), it is straightforward to compute that \(\partial F/\partial z = 2z = 0\). Therefore, at such a point, the Implicit Function Theorem does not apply. It is hardly surprising since it is clear from its graph that \(z\) is not determined as a function of \((x, y)\) in an open set about the point \((x, y) = (0, 2)\).

On the other hand, at any other point, for instance, at \((x, y, z) = (1, 1, \sqrt{2})\), the condition \(\partial F/\partial z \neq 0\) is fulfilled and thus the Implicit Function Theorem applies to state that there is a function \(f\) defined in a neighborhood of \((1, 1)\) such that \(f(1, 1) = \sqrt{2}\) and \(f\) satisfies the equation \(F(x, y, f(x, y)) = 0\) identically. Indeed, for any point \((x, y, z)\) on the sphere with \(z > 0\), the function \(f(x, y) = (4 - x^2 - y^2)^{1/2}\) will do the job and for a point \((x, y, z)\) with \(z < 0\) the function \(f(x, y) = -(4 - x^2 - y^2)^{1/2}\) is the choice.

Note that at the point \((0, 2, 0)\) we do have the condition \(\partial F/\partial y \neq 0\), and thus the Implicit Function Theorem implies that \(y\) is determined as a function of \((x, z)\) near this point.

Let us now consider a more general case discussed [Apo, pp. 296–298]. Let us consider the system

\[
F(x, y, z, w) = 0 \quad \text{and} \quad G(x, y, z, w) = 0,
\]

where \((F, G) : \mathbb{R}^4 \to \mathbb{R}^2\) is of class \(C^1\). (We have inserted an extra variable to make things more interesting.) Assuming there are functions \(x = X(z, w)\) and \(y = Y(z, w)\) which satisfies the system for all points in an open set in the \((z, w)\)-plane, we have

\[
F(X(z, w), Y(z, w), z, w) = 0 \quad \text{and} \quad G(X(z, w), Y(z, w), z, w) = 0.
\]

By the chain rule, then it follows that

\[
\frac{\partial F}{\partial x} \frac{\partial X}{\partial z} + \frac{\partial F}{\partial y} \frac{\partial Y}{\partial z} + \frac{\partial F}{\partial z} = 0
\]

and

\[
\frac{\partial G}{\partial x} \frac{\partial X}{\partial z} + \frac{\partial G}{\partial y} \frac{\partial Y}{\partial z} + \frac{\partial G}{\partial z} = 0
\]

We may view these as a system of linear equations for \(\partial X/\partial z\) and \(\partial Y/\partial z\) as

\[
\begin{bmatrix}
\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\
\frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial X}{\partial z} \\
\frac{\partial Y}{\partial z}
\end{bmatrix}
= -\begin{bmatrix}
\frac{\partial F}{\partial z} \\
\frac{\partial G}{\partial z}
\end{bmatrix}.
\]

Note that the coefficient matrix is the Jacobian matrix \(\frac{\partial (F, G)}{\partial (x, y)}\). Since it is known that the system is solvable, we may write the solution as

\[
\begin{bmatrix}
\frac{\partial X}{\partial z} \\
\frac{\partial Y}{\partial z}
\end{bmatrix}
= -\left(\frac{\partial (F, G)}{\partial (x, y)}\right)^{-1}
\begin{bmatrix}
\frac{\partial F}{\partial z} \\
\frac{\partial G}{\partial z}
\end{bmatrix}.
\]

Or, one may use Cramer’s rule to write the solution as in (9.24) in [Apo].

Let us now note that it is necessary to carry out these calculations that \(\frac{\partial (F, G)}{\partial (x, y)}\) is non-singular. Again, it is surprising that it is also sufficient to justify the assumptions we have made.
The Implicit Function Theorem. We now state the Implicit Function Theorem. This important theorem gives a condition under which one can locally solve an equation of the form $F(x, y) = 0$ for $x$ in terms of $y$. Geometrically, the solution locus of points $(x, y)$ satisfying the equation is represented as a graph of a function $x = f(y)$.

**Theorem 2** (Implicit Function Theorem). Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^{n}$ be a function of class $C^1$; we may write $F(x, y) = F(x_1, \ldots, x_n, y_1, \ldots, y_m)$. If $F(x_0, y_0) = 0$ and

$$
\frac{\partial (F_1, F_2, \ldots, F_n)}{\partial (x_1, x_2, \ldots, x_n)}(x_0, y_0) = \begin{bmatrix}
\frac{\partial F_1}{\partial x_1}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial x_n}(x_0, y_0) \\
\frac{\partial F_2}{\partial x_1}(x_0, y_0) & \cdots & \frac{\partial F_2}{\partial x_n}(x_0, y_0) \\
\vdots & & \vdots \\
\frac{\partial F_n}{\partial x_1}(x_0, y_0) & \cdots & \frac{\partial F_n}{\partial x_n}(x_0, y_0)
\end{bmatrix}
$$

is non-singular*, then there exists a unique differentiable function $f : B \subset \mathbb{R}^m \to \mathbb{R}^n$ with $y_0 \in B$ such that $f(y_0) = x_0$ and $F(f(y), y) = 0$ for all $y \in B$.

The proof is beyond the scope of this course, and it is deferred to 18.100B, for instance. Below, we shall prove the case when $F : \mathbb{R}^3 \to \mathbb{R}$.

**Remark 3** (Historical Notes). The Implicit Function Theorem is, along with its cousin the Inverse Function Theorem, one of the most important result of modern analysis. The historical evolution since Newton of its statement as well as its proofs is of great interest. The Implicit Function Theorem is valid not only for functions of several real-variables but also in more general settings of Banach spaces and manifolds. One of the most powerful forms of the Implicit Function Theorem is one due to John Nash and Jürgen Moser†. It was first used by Nash to prove his celebrated imbedding theorem for Riemannian manifolds and subsequent applications have been found in many problems in partial differential equations, geometry and other fields‡. For a nice historical recollection on the Implicit Function Theorem and for a proof of a version of the Nash-Moser theorem, I recommend [KrPa].

**Proof of the Implicit Function Theorem for $F : \mathbb{R}^3 \to \mathbb{R}$.** Denoting points in $\mathbb{R}^3$ by $(x, z)$, where $x = (x, y)$ and $z \in \mathbb{R}$, we assume that $x_0 = (x_0, y_0)$ and $(x_0, z_0)$ satisfies $F(x_0, z_0) = 0$ and $\partial F/\partial z(x_0, z_0) \neq 0$ so that it is either positive or negative. Suppose for definiteness that it is positive.

By continuity, we can find numbers $a > 0$ and $b > 0$ such that if $||x - x_0|| < a$ and $|z - z_0| < a$ then $\partial F/\partial z(x, z) > b > 0$. We can also assume that $|\partial F/\partial x(x, z)| \leq M$ and $|\partial F/\partial y(x, z)| \leq M$ for some $M > 0$ in this region. Write $F(x, z)$ as

$$
F(x, z) = F(x, z) - F(x_0, z_0) = (F(x, z) - F(x_0, z)) - (F(x_0, z) - F(x_0, z_0)).
$$

Consider the function

$$
h(t) = F(tx + (1 - t)x_0, z)
$$

for fixed $x$ and $z$. By the mean value theorem, there is a number $\theta$ between 0 and 1 such that

$$
h(1) - h(0) = h'(\theta)(1 - 0) = h'(\theta),
$$

that is, $\theta$ is such that

$$
F(x, z) - F(x_0, z) = D_x F(\theta x + (1 - \theta)x_0, z)(x - x_0).
$$

---

*Note that it is an $n \times n$ matrix.

†This version is often referred to as the Nash-Moser Theorem.

‡I have written a research article on a version of the Nash-Moser theorem to find solitary water-waves of Korteweg-de Vries type.
Thus, from (2) and the choice of $b$, it follows that $\|x - x_0\| < b$ implies that $F(x, z_0 + a_0) > 0$ and $F(x, z_0 - a_0) < 0$. Thus, by the intermediate value theorem applied to $F(x, z)$ as a function of $z$ for each $x$ there is a $z$ between $z_0 - a_0$ and $z_0 + a_0$ such that $F(x, z) = 0$. This $z$ is unique, since, by elementary calculus, a function with a positive derivative is strictly increasing and thus can have no more than one zero.

Let $U$ be the open ball of radius $\delta$ and center $x_0$ in $\mathbb{R}^2$. We have proved that if $x$ is confined to $B$, there is a unique $z$ in the interval $(z_0 - a_0, z_0 + a_0)$ such that $F(x, z) = 0$. This defines the function $z = f(x) = f(x, y)$ required by the theorem. We leave it as an exercise to prove from this construction that $z = f(x, y)$ is continuous.

It remains to prove the continuous differentiability of $z = f(x)$. From (2) and since $F(x, z) = 0$ and $z_0 = f(x_0)$ we have

$$f(x) - f(x_0) = -\frac{D_x F(\theta x + (1 - \theta) x_0, z) (x - x_0)}{\partial F / \partial z(x_0, \phi z + (1 - \phi) z_0)}.$$

If we let $x = (x_0 + h, y_0)$ then this equation becomes

$$\frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = -\frac{\partial F / \partial x(\theta x + (1 - \theta) x_0, z)}{\partial F / \partial z(x_0, \phi z + (1 - \phi) z_0)}.$$

As $h \to 0$, it follows that $x \to x_0$ and $z \to z_0$ and so we get

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = -\frac{\partial F / \partial x(x_0, z)}{\partial F / \partial z(x_0, z)}.$$

Therefore,

$$\frac{\partial f}{\partial x}(x_0, y_0) = -\frac{\partial F / \partial x(x_0, z)}{\partial F / \partial z(x_0, z)}.$$

\[\square\]

**Example 4.** Let us consider the equations

$$F(x, y, z, w) = 3x^2 z + 6wy^2 - 2z + 1 = 0, \quad G(x, y, z, w) = xz - 4y/z - 3w - z = 0.$$

It is readily seen that $(1, 2, -1, 0)$ satisfies both equations. We calculate that

$$\frac{\partial (F, G)}{\partial (x, y)} = \begin{bmatrix} 6xz & 12wy \\ z & -4/z \end{bmatrix}.$$

Evaluated at $(1, 2, -1, 0)$, this becomes $\begin{bmatrix} -6 & 0 \\ -1 & 4 \end{bmatrix}$, which is non-singular. Indeed, its determinant is -24. Therefore, according to the implicit function theorem, there exists unique differentiable functions $x = X(z, w)$ and $y = Y(z, w)$, defined in a neighborhood of $(-1, 0)$ in the $(z, w)$-plane,
satisfy these equations identically, such that \( X(-1, 0) = 1 \) and \( Y(-1, 0) = 2 \). We may compute the partial derivatives \( \partial X / \partial z \) and \( \partial Y / \partial z \) at this point. It is left as a pset problem.

On the other hand, at the point \( (1, 1/2, 2, -2) \) the implicit function theorem does not apply. It is also left as a pset problem.

REFERENCES


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