Cardinalities. Recall that the cardinal number $|A|$ of a finite set A (to be thought of as the number of elements in A) is the unique n such that there is a bijection $A \to \{1, \dots, n\}$, equivalently, such that there is a bijection $A \to \{0, \dots, n-1\}$. In fact, in the generally accepted axiomatic Set Theory used as a foundation for all of Mathematics, the natural number n is simply the set $\{0, \dots, n-1\}$ of all smaller natural numbers, and is formally introduced by means of a recursive definition:

$$0 \ := \ \emptyset, \quad 1 \ := \ \{0\}, \quad 2 \ := \ \{0,1\}, \quad \text{and in general,} \ \ n+1 \ := \ n \cup \{n\}.$$

(For details and proofs of various facts stated here, see Halmos' book "Naive Set Theory", or Moschovakis' undergraduate text for a very thorough treatment.)

More generally, in Set Theory one singles out certain sets as being *cardinal numbers* (also called cardinals, or cardinalities). For example, every natural number $n = \{0, \dots, n-1\}$ is a cardinal, but so is the set $\mathbb{N}$ of all natural numbers. (In its role as a cardinal, we denote $\mathbb{N}$ by $\aleph_0$, the subscript 0 indicating that it is the least infinite cardinal, as defined later.) The key fact about cardinals is that for every set A there is a unique cardinal κ for which there is a bijection $A \to \kappa$; this κ is called the *cardinality of* A (to be thought of as the number of elements of S) and denoted by $|A|$. Thus $|\kappa| = \kappa$ and for any sets A and B we have:

$$|A| \ = \ |B| \iff \text{there is a bijection} \ \ A \to B.$$

Below we let $\kappa, \kappa_1, \kappa_2, \dots$ range over cardinals. For example, $|A| = \aleph_0$ is equivalent to A being countably infinite. The cardinals n are said to be *finite*, all other ones are said to be *infinite*. We define $\kappa_1 \le \kappa_2$ to mean that there exists an injective map $\kappa_1 \to \kappa_2$. For $\kappa_1 = m, \kappa_2 = n$ this is equivalent to the usual meaning of $m \le n$, which in turn is equivalent to the purely set-theoretic condition $m \subseteq n$. More generally, one can show that for all κ_1, κ_2 we have

$$\kappa_2 \ \le \ \kappa_2 \iff \kappa_1 \subseteq \kappa_2.$$

It is easy to verify using the definition or the equivalence just mentioned that $\kappa \le \kappa$, and $\kappa_1 \le \kappa_2, \ \kappa_2 \le \kappa_3 \Rightarrow \kappa_1 \le \kappa_3$. We also have antisymmetry:

$$\kappa_1 \le \kappa_2, \ \kappa_2 \le \kappa_1 \implies \kappa_1 = \kappa_2,$$

but this is less easy, and is called the *Schröder-Bernstein Theorem*. We even have linearity, but this is also non-trivial:

$$\kappa_1 \le \kappa_2 \ \text{ or } \ \kappa_2 \le \kappa_1.$$

Thus the class of cardinals is linearly ordered by $\le$. (We say "class", because one can show that there is no set with every cardinal as an element: there are too many cardinals to form a set.) Of course, $\kappa_1 < \kappa_2$ stands for: $\kappa_1 \le \kappa_2$ and $\kappa_1 \ne \kappa_2$.

For example we have $n < \aleph_0 \le \kappa$ for all infinite κ. For every cardinal there is a greater cardinal, even a least greater one: $\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_{\aleph_0} < \cdots$.

Cardinals can be added and multiplied: if $\kappa_1 = |A_1|$ and $\kappa_2 = |A_2|$, then

$$\kappa_1 + \kappa_2 = |A_1 \cup A_2| \text{ when } A_1 \cap A_2 = \emptyset; \qquad \kappa_2 \times \kappa_2 = |A_1 \times A_2|,$$

and these operations are easily seen to be commutative and associative. Actually, for infinite κ these operations behave in a much simpler way than addition and multiplication of natural numbers: if κ_1 or κ_2 is infinite, then $\kappa_1 + \kappa_2 = \max(\kappa_1, \kappa_2)$, and if $\kappa_1 \ne 0$ and κ_2 is infinite, then also $\kappa_1 \times \kappa_2 = \max(\kappa_1, \kappa_2)$.

Recall that a union of countably many countable sets is countable. This is the case $\kappa = \aleph_0$ of the following more general result:

Fact: Suppose κ is an infinite cardinal and $(A_i)_{i\in I}$ is a family of sets with $|I| \le \kappa$ and $|A_i| \le \kappa$ for all $i \in I$. Then $|\bigcup_{i\in I} A_i| \le \kappa$.

Idea of proof: intuitively (this can be made precise) one expects $|\bigcup_{i\in I} A_i| \le \kappa \times \kappa$. Now use that $\kappa \times \kappa = \kappa$.

Corollary 0.1. *Let κ be an infinite cardinal and $|A| \le \kappa$. Then $|A^*| \le \kappa$.*

Proof. Recall that $A^* = \bigcup_n A^n$. Now $|A^0| = 1 \le \kappa$ and $|A^1| = |A| \le \kappa$. Assuming inductively that $|A^n| \le \kappa$, we use an obvious bijection $A^{n+1} \to A^n \times A$ to get $|A^{n+1}| = |A^n \times A| \le \kappa \times \kappa = \kappa$. Thus we have shown by induction that $|A^n| \le \kappa$ for all n. By the **Fact** above this gives $|A^*| \le \kappa$. □

We wish to apply this last result to prove a somewhat more precise version of the upward Löwenheim-Skolem theorem. Let L be a language (a set of relation symbols and function symbols).

Lemma 0.2. *Let κ be an infinite cardinal, and suppose $|L| \le \kappa$. Then*

$$|\text{set of } L\text{-terms}| \ \le \ \kappa, \qquad |\text{set of } L\text{-formulas}| \ \le \ \kappa.$$

Proof. Use that L-terms and L-formulas are words on the alphabet

$$A \ := \ L \cup \{x_0, x_1, x_2, \dots\} \cup \{=, \text{connectives}, \forall, \exists\} \cup \{(,,,)\}.$$

Now apply the previous corollary to the fact that $|A| \le \kappa$. □

For $\kappa = \aleph_0$ the following is the "downward" Löwenheim-Skolem theorem:

Lemma 0.3. *Let κ be an infinite cardinal, suppose $|L| \le \kappa$ and T is a consistent L-theory. Then T has a model $\mathcal{A} = \langle A; \dots \rangle$ of cardinality $|A| \le \kappa$.*

Proof. This is proved in the same way as the downward Löwenheim-Skolem theorem: in that proof we construct a model of T whose elements are equivalence classes of variable-free L^*-terms, for a certain extended language $L^* = \bigcup L_n$, with $L = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$. The key thing to check by induction on n and with the help of the lemma above is that $|L_n| \le \kappa$ for all n. Then by the **Fact** above we also have $|L^*| \le \kappa$, and thus, using again the lemma above, the model constructed will have cardinality $\le \kappa$. □

This leads to a more precise form of the upward Löwenheim-Skolem theorem:

Theorem 0.4. *Let κ be an infinite cardinal, suppose $|L| \le \kappa$ and T is an L-theory with an infinite model. Then T has a model $\mathcal{B} = \langle B; \dots \rangle$ of cardinality $|B| = \kappa$.*

Proof. Take a set A with $|A| = \kappa$. Let $L_A := L \cup \{c_a : a \in A\}$ where c_a is for every $a \in A$ a constant symbol not in L and $c_a \ne c_b$ whenever $a, b \in A$ and $a \ne b$. This extended language L_A has cardinality $|L_A| = \kappa$. Set

$$\Gamma \ := \ T \cup \{c_a \ne c_b : \ a, b \in A, \ a \ne b\},$$

a set of L_A-sentences. The assumption that T has an infinite model yields that every finite subset of Γ has a model. Thus by compactness, Γ has a model, and any such model must have cardinality $\ge \kappa$. But the lemma above gives a model of cardinality $\le \kappa$, so this model must have cardinality equal to κ. Then the L-reduct of this model is a model of T of cardinality κ. □