HW 9, due Friday, March 29: problems 1 and 7 on p. 96 of the book (where x, y, z are distinct variables). For the second part of problem 1 and of problem 7 you can use the following lemma to the effect that we can change bound variables. This lemma is really the correct formulation of 1(iii) on p. 90.

Lemma 0.1. *Assume the variable v does not occur in ϕ. Then*

$$\forall x\phi \vdash \forall v\phi[v/x]$$

Proof. From $\forall x\phi$ as hypothesis we get $\phi[v/x]$ by ∀E, and then $\forall v\phi[v/x]$ by ∀I. (This last step is allowed since v does not occur in the hypothesis $\forall x\phi$ on which $\phi[v/x]$ depends.) □

Solution of 1. Taking as hypotheses $\forall x(x = x)$ and $\forall xyz(x = y \wedge z = y) \to x = z)$ we have to derive $I_2 \wedge I_3$. We shall derive I_2 and I_3 separately from these hypotheses; then using ∧I we obtain $I_2 \wedge I_3$. Note that I_2 is $\forall xy(x = y \to y = x)$. From the second hypothesis we get $\forall yz(y = y \wedge z = y) \to y = z)$ by ∀E and the substitution $[y/x]$. Using ∀E again we get $\forall z(y = y \wedge z = y) \to y = z)$. Then the substitution $[x/z]$ and ∀E gives $(y = y \wedge x = y) \to y = x)$, which by the rules of propositional logic, (see Theorem 2.4.4 (6) yields $y = y \to (x = y \to y = x)$.

From the first hypothesis $\forall x(x = x)$ we get $y = y$ by ∀E and the substitution $[y/x]$. This gives $y = x \to x = y$ by →I; then using ∀I twice we get I_2.

I_3 is $\forall xyz(x = y \wedge y = z \to x = z)$. Start with assuming temporarily $x = y \wedge y = z$ as a hypothesis; we will cancel it later. Then ∧E gives $y = z$. We have already derived I_2, and this yields $y = z \to z = y$ by several applications of the lemma above: change x, y to new variables u, v (different from x, y, z, and then change these to y, z. Then →E gives $z = y$. From the same temporary hypothesis $x = y \wedge y = z$ we also get $x = y$ by ∧E. Then we get $x = y \wedge z = y$ by ∧I. From the (genuine) second hypothesis we obtain $x = y \wedge z = y \to x = z$ by three applications of ∀E. Then we get $x = z$ by →E, and so we get $(x = y \wedge y = z) \to x = z$ by →I, cancelling the temporary hypothesis. Next we use ∀I three times to get I_3 as conclusion.

Solution of 7. (i) From the hypothesis $\forall xy(f(x) = f(y) \to x = y)$ we get

$$f(g(x)) = f(g(y)) \to g(x) = g(y)$$

by two applications of ∀E. Combining this with the temporary hypothesis $f(g(x)) = f(g(y))$ we get $g(x) = g(y)$ by →E. From the hypothesis $\forall xy(g(x) = g(y) \to x = y)$ we get $g(x) = g(y) \to x = y$ by two applications of ∀E. Then →E gives $x = y$, and so $f(g(x)) = f(g(y)) \to x = y$ by →I, canceling the hypothesis $f(g(x)) = f(g(y))$. Two applications of ∀I then yield the desired conclusion

$$\forall xy(f(g(x)) = f(g(y)) \to x = y).$$

Note that this derivation expresses a special case of the fact that the composition of two injective functions is injective (the special case where all domains and codomains are the same nonempty set).

(ii) This expresses the fact that the composition of two surjective functions with all domains and codomains the same nonempty set is surjective. It is convenient to change some of the bound variables in the hypotheses. Let u be a variable different from x, y, z. From the first hypothesis we get $\exists x f(x) = y$ by $\forall E$, from which we can derive $\exists u(f(u) = y$. From the second hypothesis we get $\exists x(g(x) = u)$ by ∀E.