HW 7, due Monday, March 11: problems 12 and 14(b) on p. 76 of the book. (For Friday March 8, do some other problems on that page: most of you will benefit.)

Solution of 12, p. 76. We have to show that for every L-structure $\mathcal{A}$,

$$\mathcal{A} \models \neg\exists y \forall x \big(S(y,x) \leftrightarrow \neg S(x,x)\big).$$

Let $\mathcal{A} = \langle A; S, \dots \rangle$ be an L-structure and suppose towards a contradiction that $\mathcal{A} \not\models \neg\exists y \forall x \big(S(y,x) \leftrightarrow \neg S(x,x)\big)$. Then $\mathcal{A} \models \exists y \forall x \big(S(y,x) \leftrightarrow \neg S(x,x)\big)$, so we have $b \in A$ such that for all $a \in A$, $S(b,a)$ iff not $S(a,a)$: think of b as a barber in a village who shaves exactly those adult males of the village who don't shave themselves. We assume here tacitly that the barber is itself an adult male, so we can apply this for $a = b$ to get: $S(b,b)$ iff not $S(b,b)$, which gives a contradiction, since either $S(b,b)$, or not $S(b,b)$, and we cannot have both. (Thus we have shown that there cannot exist villages with a barber shaving exactly those adult males who do not shave themselves).

Solution of 14(b). Here ϕ and ψ are assumed to be quantifier-free. First, take a variable y different from x that does not occur in ϕ or ψ. Then $\forall x\phi \leftrightarrow \exists x\psi$ has the variant $\forall x\phi \leftrightarrow \exists y\psi'$ with $\psi' := \psi[y/x]$. Next, eliminating $\leftrightarrow$ in the usual way, $\forall x\phi \leftrightarrow \exists y\psi'$ transforms into

$$(\neg\forall x\phi \vee \exists y\psi') \wedge (\neg\exists y\psi' \vee \forall x\phi)$$

which by a few prenex transformations is transformed into

$$(\exists x\neg\phi \vee \exists y\psi') \wedge (\forall y\neg\psi' \vee \forall x\phi),$$

which by further prenex transformations changes into

$$\exists x(\neg\phi \vee \exists y\psi') \wedge \forall y(\neg\psi' \vee \forall x\phi),$$

and by two more such transformations becomes

$$\exists x\exists y(\neg\phi \vee \psi') \wedge \forall y\forall x(\neg\psi' \vee \phi).$$

At this stage we need to change again bounded variables: take variables $u, v \notin \{x, y\}$ that do not occur in ϕ or in ψ'. Set $\theta := (\neg\psi' \vee \phi)[u, v/y, x]$. Then the formula in the last display has the variant

$$\exists x\exists y(\neg\phi \vee \psi') \wedge \forall u\forall v\theta,$$

and one prenex transformation transforms this into

$$\exists x\big[\exists y(\neg\phi \vee \psi') \wedge \forall u\forall v\theta\big],$$

which by successive further prenex transformations yields first

$$\exists x\exists y[(\neg\phi \vee \psi') \wedge \forall u\forall v\theta],$$

next $\exists x\exists y\forall u[(\neg\phi \vee \psi') \wedge \forall v\theta\big]$, and finally

$$\exists x\exists y\forall u\forall v[(\neg\phi \vee \psi') \wedge \theta],$$

which is in prenex form.