**HW 6**, due Monday, March 4: problems 1 and 5 on p. 68 of the book. (For Friday, March 1, problem 1, (i)–(vii) on p.56 and 4 on p. 63, but don't hand in.)

**Solution of 1, p. 68**. Note that here $N$ is our $\mathbb{N} = \{0, 1, 2, \dots\}$, and that in our notation system the similarity type is $\langle -; 2, 2, 1, 0\rangle$, since we regard constants as nullary functions. To make things more readable we let $+, -, S, 0$ also denote the function symbols of the language $L$ that name the operations on $\mathbb{N}$ that are usually written with those symbols, and we use infix notation: $t_1 + t_2$ and $t_1 t_2$, rather than the official prefix notation: $+(t_1, t_2)$ and $\cdot(t_1, t_2)$; we also drop parentheses whenever it is clear from the context where they should be inserted in the official notation.

(i) $(2+2)+1$, $(2+2)+(1+0)$, $2+(2+1)$, $(2+1)+2$, and $SSSSS0$ are 5 distinct closed $L$-terms $t$ with $t^{\mathcal{N}} = 5$.

(ii) By induction on $n$. For $n = 0$ the term 0 does the job. Assume the closed $L$-term $t$ is such that $t^{\mathcal{N}} = n$. Then $S(t)$ is a closed $L$-term such that $S(t)^{\mathcal{N}} = n + 1$.

(iii) Use (ii) and the fact that if $t$ is a closed $L$-term with $t^{\mathcal{N}} = n$, then $t + 0$ is also a closed $L$-term with $(t + 0)^{\mathcal{N}} = n$, and $t + 0$ has greater length (as a word on the alphabet $L$) than $t$.

**Solution of 5, p. 68**. Take any $L$-structure $\mathcal{A} = \langle A; \dots\rangle$ such that $A$ has more than one element. Let $\phi$ be the $L$-formula $x_0 = x_1$. Take two distinct elements $a$ and $b$ of $A$. Then $a \neq b$ gives $\mathcal{A} \not\models \phi[a, b/x_0, x_1]$, so $\mathcal{A} \not\models \phi$. On the other hand, $\mathcal{A} \models \phi[a, a/x_0, x_1]$, so $\mathcal{A} \not\models \neg\phi$.

Next, let $\sigma$ be the $L$-sentence $\forall x_0 \forall x_1 (x_0 = x_1)$. Then $\sigma$ is false in the above $\mathcal{A}$, so $\not\models \sigma$. But $\sigma$ is true in some $L$-structure, namely any $L$-structure that has just one element. Thus $\not\models \neg\sigma$.