HW 6, due Monday, March 4: problems 1 and 5 on p. 68 of the book. (For Friday, March 1, problem 1, (i)-(vii) on p. 56 and 4 on p. 63, but don’t hand in.)

Solution of 1, p. 68. Note that here \( N \) is our \( N = \{0, 1, 2, \ldots\} \), and that in our notation system the similarity type is \( \langle -, 2, 2, 1, 0 \rangle \), since we regard constants as nullary functions. To make things more readable we let \(+, -, S, 0\) also denote the function symbols of the language \( L \) that name the operations on \( N \) that are usually written with those symbols, and we use infix notation: \( t_1 + t_2 \) and \( t_1 t_2 \), rather than the official prefix notation: \( +(t_1, t_2) \) and \( \cdot(t_1, t_2) \); we also drop parentheses whenever it is clear from the context where they should be inserted in the official notation.

(i) \( (2 + 2) + 1, (2 + 2) + (1 + 0), 2 + (2 + 1), (2 + 1) + 2, \) and \( SSSS0 \) are 5 distinct closed \( L \)-terms \( t \) with \( t^N = 5 \).

(ii) By induction on \( n \). For \( n = 0 \) the term 0 does the job. Assume the closed \( L \)-term \( t \) is such that \( t^N = n \). Then \( S(t) \) is a closed \( L \)-term such that \( S(t)^N = n + 1 \).

(iii) Use (ii) and the fact that if \( t \) is a closed \( L \)-term with \( t^N = n \), then \( t + 0 \) is also a closed \( L \)-term with \( (t + 0)^N = n \), and \( t + 0 \) has greater length (as a word on the alphabet \( L \)) than \( t \).

Solution of 5, p. 68. Take any \( L \)-structure \( \mathcal{A} = \langle A; \ldots \rangle \) such that \( A \) has more than one element. Let \( \phi \) be the \( L \)-formula \( x_0 = x_1 \). Take two distinct elements \( a \) and \( b \) of \( A \). Then \( a \neq b \) gives \( \mathcal{A} \not\models \phi[a, b, x_0, x_1] \), so \( \mathcal{A} \not\models \phi \). On the other hand, \( \mathcal{A} \models \phi[a, a, x_0, x_1] \), so \( \mathcal{A} \not\models \neg \phi \).

Next, let \( \sigma \) be the \( L \)-sentence \( \forall x_0 \forall x_1 (x_0 = x_1) \). Then \( \sigma \) is false in the above \( \mathcal{A} \), so \( \not\models \sigma \). But \( \sigma \) is true in some \( L \)-structure, namely any \( L \)-structure that has just one element. Thus \( \not\models \neg \sigma \).