HW 4, due Monday, February 11: problems 4 and 5 on p. 37 of the book. (For Friday, February 8, do the earlier problems 1 and 3 on that page, but don't hand them in.)

Solution of 4. For the first one, take as hypotheses

$$(\phi \to \psi) \to (\phi \to \sigma), \quad \psi$$

From hypothesis ψ we get $\phi \to \psi$ by $\to I$ without canceling any hypothesis. Using the first hypothesis this gives $\phi \to \sigma$ by $\to E$. At this point we add ϕ as a hypothesis, and get σ by $\to E$. Canceling the hypothesis ψ we get $\psi \to \sigma$ by $\to I$, and then another use of $\to I$ gives $\phi \to (\psi \to \sigma$, while canceling the hypothesis ϕ. At this point we can cancel the first hypothesis to get

$$[(\phi \to \psi) \to (\phi \to \sigma)] \to [\phi \to (\psi \to \sigma)]$$

using $\to I$.

The second one is a bit tricky, and we only give a sketch. The idea is to use the result of 1(d) to reduce to showing $\vdash \neg[\{(\phi \to \psi) \to \phi\} \wedge \neg\phi]$. To get that, we take as hypothesis $\neg\neg[\{(\phi \to \psi) \to \phi\} \wedge \neg\phi]$ and try to derive from it $\perp$; if we succeed we can cancel this hypothesis using RAA to get $\neg[\{(\phi \to \psi) \to \phi\} \wedge \neg\phi]$, which by 1(d) leads easily to the desired conclusion.

From the hypothesis $\neg\neg[\{(\phi \to \psi) \to \phi\} \wedge \neg\phi]$ we first derive $\{(\phi \to \psi) \to \phi\} \wedge \neg\phi$ using Theorem 2.4.4 (5). Using $\wedge E$ we then get $(\phi \to \psi) \to \phi$ and $\neg\phi$. From $(\phi \to \psi) \to \phi$ and using Theorem 2.4.4 (4) we derive $\neg\phi \to \neg(\phi \to \psi)$. We already have $\neg\phi$, so we can apply $\to E$ to get $\neg(\phi \to \psi)$, which by 1(d) leads to $\phi \wedge \neg\psi$, and thus to ϕ by $\wedge E$. Since we also have $\neg\phi$, we then get $\perp$ by $\to E$.

Solution of 5. The first claim of 5 follows from how we defined the concept "$\Gamma \vdash \phi$". To prove the second statement, assume $\Gamma \vdash \phi$ and $\Delta, \phi \vdash \psi$.

Start with a derivation witnessing $\Gamma \vdash \phi$, that is, a derivation of ϕ all whose uncancelled hypotheses are in Γ. Attaching to this a derivation witnessing $\phi, \Delta \vdash \psi$, we obtain a derivation of ψ all whose uncancelled hypotheses are in $\Gamma \cup \Delta$, that is, witnessing $\Gamma \cup \Delta \vdash \psi$.