HW 3, due Monday, February 4: problems 3 and 4 on p. 27 of the book. (For Wednesday and Friday, January 30 and February 1, do the earlier problems 1, 2 on p. 20, and 1, 2 on p. 27, but don’t hand them in.)

General Convention: $\phi(p_0, \ldots, p_n)$ denotes a proposition $\phi$ in which no $p_i$ with $i > n$ occurs.

From the book we learn that an $n$-ary connective has an associated truth table $\{0, 1\}^n \to \{0, 1\}$. Let $S$ be a set of connectives, and let $\text{PROP}(S)$ be the set of propositions built up using these connectives and $p_0, p_1, \ldots$. To say that $S$ is functionally complete means that for every $n$, every $\phi(p_0, \ldots, p_n) \in \text{PROP}$ is semantically equivalent to some $\psi(p_0, \ldots, p_n) \in \text{PROP}(S)$.

Solution of 3, p. 27. Suppose we only allow the connective $\neg$. Then the propositions $\phi(p_0)$ (that is, using only the variable $p_0$) are: $p_0, \neg p_0, \neg\neg p_0, \neg\neg\neg p_0, \ldots$, which are all semantically equivalent to $p_0$ or to $\neg p_0$. Thus no $\phi(p_0)$ is semantically equivalent to $p_0 \land \neg p_0$, hence $\{\neg\}$ is not functionally complete.

Next, suppose we only allow the connectives $\to, \lor$. Then it is easy to check by induction on formulas $\phi(p_0)$ that for every valuation $v$ with $v(p_0) = 1$ we have $v(\phi(p_0)) = 1$. But $v(p_0 \land \neg p_0) = 0$ for every $v$. Thus as before we see that no $\phi(p_0)$ is semantically equivalent to $p_0 \land \neg p_0$, hence $\{\to, \lor\}$ is not functionally complete.

Solution of 4, p. 27. By definition of the Sheffer stroke, $\phi|\psi$ is semantically equivalent to $(\phi \land \psi)$. Hence $\phi|\phi$ is semantically equivalent to $\neg \phi$, and thus $\neg(\phi|\psi)$ is semantically equivalent to both $\phi \land \psi$ and to $(\phi|\psi)|(\phi|\psi)$. This shows how to express $\neg$ and $\land$ in terms of $|$ alone. Since $\{\neg, \land\}$ is functionally complete, it follows that $\{|\}$ is too.