HW 3, due Monday, February 4: problems 3 and 4 on p. 27 of the book. (For Wednesday and Friday, January 30 and February 1, do the earlier problems 1,2 on p. 20, and 1,2 on p. 27, but don't hand them in.)

General Convention: $\phi(p_0, \ldots, p_n)$ denotes a proposition ϕ in which no p_i with $i > n$ occurs.

From the book we learn that an n-ary connective has an associated truth table $\{0,1\}^n \to \{0,1\}$. Let S be a set of connectives, and let PROP(S) be the set of propositions built up using these connectives and $p_0, p_1, \ldots$. To say that S is functionally complete means that for every n, every $\phi(p_0, \ldots, p_n) \in \text{PROP}$ is semantically equivalent to some $\psi(p_0, \ldots, p_n) \in \text{PROP}(S)$.

Solution of 3, p. 27. Suppose we only allow the connective $\neg$. Then the propositions $\phi(p_0)$ (that is, using only the variable p_0) are: $p_0, \neg p_0, \neg\neg p_0, \neg\neg\neg p_0, \ldots,$ which are all semantically equivalent to p_0 or to $\neg p_0$. Thus no $\phi(p_0)$ is semantically equivalent to $p_0 \wedge \neg p_0$, hence $\{\neg\}$ is not functionally complete.

Next, suppose we only allow the connectives $\to, \vee$. Then it is easy to check by induction on formulas $\phi(p_0)$ that for every valuation v with $v(p_0) = 1$ we have $v(\phi(p_0)) = 1$. But $v(p_0 \wedge \neg p_o) = 0$ for every v. Thus as before we see that no $\phi(p_0)$ is semantically equivalent to $p_0 \wedge \neg p_0$, hence $\{\to, \vee\}$ is not functionally complete.

Solution of 4, p. 27. By definition of the Sheffer stroke, $\phi|\psi$ is semantically equivalent to $\neg(\phi \wedge \psi)$. Hence $\phi|\phi$ is semantically equivalent to $\neg\phi$, and thus $\neg(\phi|\psi)$ is semantically equivalent to both $\phi \wedge \psi$ and to $(\phi|\psi)|(\phi|\psi)$. This shows how to express $\neg$ and $\wedge$ in terms of $|$ alone. Since $\{\neg, \wedge\}$ is functionally complete, it follows that $\{|\}$ is too.