HW 1

Problem 1: indicate a bijection \( \mathbb{N} \rightarrow \mathbb{Z} \).

Problem 2: indicate a bijection \( \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \).

Solution of 1: \( f : \mathbb{N} \rightarrow \mathbb{Z} \) defined by \( f(2n) = n \) and \( f(2n + 1) = -(n + 1) \) is a bijection. It corresponds to enumerating (listing) \( \mathbb{Z} \) as \( 0, -1, 1, -2, 2, -3, 3, \ldots \).

Solution of 2: order the pairs \((m, n)\) according to their sum \(m + n\), and for a given value of \(m + n\), order them according to \(m\), so we get the list

\[
(0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), (0, 3), (1, 2), (2, 1), (3, 0), (0, 4), \ldots
\]

in which every pair \((m, n)\) shows up exactly once. More precisely, the function \( f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) defined by

\[
f(m, n) := \frac{(m + n)(m + n + 1)}{2} + m
\]

is a bijection with \( f(0, 0) = 0, f(0, 1) = 1, f(1, 0) = 2, f(0, 2) = 3, f(1, 1) = 4 \) and so on. Another solution of Problem 2 uses the fact that every natural number \( \geq 1 \) is of the form \( 2^m(2n + 1) \) for a unique pair \((m, n)\). Thus the function \( g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) defined by \( g(m, n) := 2^m(2n + 1) - 1 \) is also a bijection. The first solution is special, since it is given by a “polynomial” function. (If I remember right, there is only one other bijection \( \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) given by a polynomial function.)