HW 1

Problem 1: indicate a bijection $\mathbb{N} \to \mathbb{Z}$.

Problem 2: indicate a bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Solution of 1: $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(2n) = n$ and $f(2n+1) = -(n+1)$ is a bijection. It corresponds to enumerating (listing) $\mathbb{Z}$ as $0, -1, 1, -2, 2, -3, 3, \ldots$.

Solution of 2: order the pairs (m, n) according to their sum $m+n$, and for a given value of $m+n$, order them according to m, so we get the list

$$(0,0),\ (0,1),\ (1,0),\ (0,2),\ (1,1),\ (2,0),\ (0,3),\ (1,2)\ (2,1),\ (3,0),\ (0,4), \ldots$$

in which every pair (m, n) shows up exactly once. More precisely, the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$f(m,n) \ := \ \frac{(m+n)(m+n+1)}{2} + m$$

is a bijection with $f(0,0) = 0,\ f(0,1) = 1,\ f(1,0) = 2,\ f(0,2) = 3, f(1,1) = 4$ and so on. Another solution of Problem 2 uses the fact that every natural number ≥ 1 is of the form $2^m(2n+1)$ for a unique pair (m, n). Thus the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $g(m,n) := 2^m(2n+1) - 1$ is also a bijection. The first solution is special, since it is given by a "polynomial" function. (If I remember right, there is only one other bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by a polynomial function.)