HW 12, due Monday April 22: problems 10 and 18 on pp. 110, 111 of the book. Let me add some clarifications to these problems, and then indicate the solutions.

Problem 10. Here $L = \{f\}$ with f a unary function symbol. A *permutation* of a set A is by definition a bijection $b : A \to A$. In (i) you are asked to produce an L-sentence σ_{sur} such that for all L-structures $\mathcal{A} = \langle A; f^{\mathcal{A}} \rangle$,

$$\mathcal{A} \models \sigma_{sur} \iff f^{\mathcal{A}} : A \to A \text{ is surjective.}$$

In (iv) you need to find an L-sentence σ such that (a) and (b) below hold:
(a) for all L-structures $\mathcal{A} = \langle A; \ldots \rangle$, if $\mathcal{A} \models \sigma$, then A is infinite;
(b) for every infinite set A there is a model $\mathcal{A} = \langle A; \ldots \rangle$ of σ.

Solution. Let x, y be distinct variables. In (i), let σ_{sur} be $\forall y \exists x (f(x) = y)$. In (ii), let σ_{in} be $\forall x, y (f(x) = f(y) \to x = y)$. In (iii) we can take σ_{bi} to be $\sigma_{sur} \wedge \sigma_{in}$. As to (iv), one can take σ to be $\sigma_{in} \wedge \neg \sigma_{sur}$. For (v), $\{\sigma_{bi}, \forall x (f(x) \neq x)\}$ has an infinite model, namely $\langle \mathbb{Z}; f \rangle$ where $f(k) = k + 1$ for all $k \in \mathbb{Z}$. Thus by Skolem-Löwenheim it has for each infinite cardinal κ a model $\langle A; f \rangle$ with $|A| = \kappa$. Given any infinite set B, let $\kappa := |B|$, take $\langle A; f \rangle$ as above, and take a bijection $b : A \to B$. Then $b \circ f \circ b^{-1} : B \to B$ is a permutation of B without fixed points.

Problem 18. Here k is a fixed (but arbitrary) natural number ≥ 1. Let the language L have just a binary relation symbol R. Then a graph is an L-structure $\mathcal{G} = \langle G; R \rangle$ where R is a symmetric irreflexive binary relation on the (nonempty) set G whose elements are thought of as the vertices of the graph. A *subgraph* of such $\mathcal{G}$ is a graph $\mathcal{G}_0 = \langle G_0; R \cap G_0^2 \}$ with $G_0 \subseteq G$. A k-coloring of such a graph $\mathcal{G} = \langle G; R \rangle$ is by definition a function $c : G \to \{1, \ldots, k\}$ such that $c(g) \neq c(h)$ for all pairs $(g, h) \in R$. In the statement of the problem the language L is augmented by k extra unary predicate symbols $C_1, \ldots, C_k$, to give the language $L_k = L \cup \{C_1, \ldots, C_k\}$ and it is observed that $\mathcal{G}$ has a k-coloring iff $\mathcal{G}$ can be expanded to an L_k-structure that satisfies the sentences listed there involving the new symbols $C_1, \ldots, C_k$. Please check for yourself that this is a correct observation, and then solve the problem. Hint: given $\mathcal{G}$, you might consider extending L_k further by names for the elements of G.

Solution. Let $\mathcal{G} = \langle G; R \rangle$ be a graph. Extend the language L_k to $L_k(G)$ by adding for each $g \in G$ a name $\underline{g}$. Let the set $\Sigma(G)$ consist of the following $L_k(G)$-sentences:

(1) for all elements $g, h \in G$, the sentence $\underline{g} \neq \underline{h}$ whenever $g \neq h$, the sentence $R(\underline{g}, \underline{h})$ whenever $(g, h) \in R$, and the sentence $\neg R(\underline{g}, \underline{h})$ whenever $(g, h) \notin R$.

(2) for each $g \in G$ the sentences $C_1(\underline{g}) \vee \cdots \vee C_k(\underline{g})$ and $\bigwedge_{1 \leq i < j \leq k} \neg (C_i(\underline{g}) \wedge C_j(\underline{g}))$ (expressing that g gets painted with exactly one of the k colors);

(3) for all $(g, h) \in R$ the sentence $\bigwedge_{i=1}^{k} \neg \big(C_i(\underline{g}) \wedge C_i(\underline{h})\big)$ (expressing that the adjacent vertices g and h do not get painted the same color).

Observe that $\mathcal{G}$ is k-colorable if and only if $\Sigma(G)$ has a model.

Now assume that every finite subgraph $\mathcal{G}_0 = \langle G_0; \cdots \rangle$ of $\mathcal{G}$ is k-colorable. Then for every finite nonempty subset G_0 of G the set $\Sigma(G_0)$ has a model. Since every finite subset of $\Sigma(G)$ is contained in $\Sigma(G_0)$ for some finite nonempty $G_0 \subseteq G$, it follows that every finite subset of $\Sigma(G)$ has a model. Hence by compactness, $\Sigma(G)$ has a model. Thus by the observation above, $\mathcal{G}$ is k-colorable.