HW 11, due Monday, April 15: problems 7 and 8 on p. 110 of the book.

Solution of 7. Assume the set Σ of L-sentences has only finite models. We have to show that for some n all models of Σ have at most n elements. Suppose towards a contradiction that Σ has for every n a model with more than n elements. Then every finite subset of $\Sigma \cup \{\lambda_n : n = 1, 2, 3, \dots\}$ has a model, and thus $\Sigma \cup \{\lambda_n : n = 1, 2, 3, \dots\}$ has a model, and such a model is an infinite model of Σ, contradicting the assumption.

Solution of 8. Let $\mathcal{A} = \langle A; P, \dots \rangle$ be a model of σ; our job is to show that A is infinite. Pick an element $a_0 \in A$. Since $\mathcal{A} \models \forall x \exists y P(x, y)$ we can take $a_1 \in A$ such that $(a_0, a_1) \in P$. Using again that $\mathcal{A} \models \forall x \exists y P(x, y)$ we pick $a_2 \in A$ such that $(a_1, a_2) \in P$. Continuing this way we obtain an infinite sequence $a_0, a_1, a_2, a_3, \dots$ in A such that $(a_n, a_{n+1}) \in P$ for all n. Since P is transitive, it follows that $(a_m, a_n) \in P$ for all $m < n$, and thus $a_m \neq a_n$ for all $m < n$. Thus A is infinite.