HW 10, due Monday, April 8: problems 1 and 2 on p. 104 of the book. In problem 2, all T_i are L-theories, for a fixed language L.

Solutions. Problem 1: The given group $\mathcal{A} = \langle A;\ e,\ ^{-1}, \cdot\rangle$ is assumed to be nontrivial, that is, $\mathcal{A} \models \exists x(x \neq e)$. Suppose T were a Henkin theory. Then necessarily $\mathcal{A} \models \exists x(x \neq e) \to e \neq e$, since e is the only constant symbol in the language of groups. Hence $\mathcal{A} \models e \neq e$, a contradiction. So T is not a Henkin theory.

Problem 2: Slightly more generally, let L be a language and let for each $i \in I$ a set Γ_i of L-sentences be given such that for all $i, j \in I$, $\Gamma_i \subseteq \Gamma_j$ or $\Gamma_j \subseteq \Gamma_i$. Set $\Gamma := \bigcup_{i \in I} \Gamma_i$. Then we claim for any L-sentence σ:

$$\Gamma \vdash \sigma \iff \Gamma_i \vdash \sigma \text{ for some } i \in I.$$

The direction $\Leftarrow$ is clear. For the direction $\Rightarrow$, assume $\Gamma \vdash \sigma$. Take a derivation of σ from Γ, and let the uncancelled hypotheses $\sigma_1, \ldots, \sigma_n \in \Gamma$ in this derivation lie in $\Gamma_{i_1}, \ldots, \Gamma_{i_n}$, respectively, where $i_1, \ldots, i_n \in I$. Among these $i_1, \ldots, i_n$ there is one, call it i, such that $\Gamma_{i_1}, \ldots, \Gamma_{i_n} \subseteq \Gamma_i$. Then this derivation is a derivation of σ from Γ_i, so $\Gamma_i \vdash \sigma$.

Taking $\sigma = \perp$ in the above, it follows that if all Γ_i are consistent, then so is Γ. Finally, take $\Gamma_i := T_i$. Then $T = \Gamma$ and it remains to show that T is a theory, so assume $T \vdash \sigma$. Then by the claim above, $T_i \vdash \sigma$ for some $i \in I$, so $\sigma \in T_i$ for such i, and thus $\sigma \in T$.