Midterm Math 414 (Logic), April 1, 2019

(1) Which of the following two statements are correct (and why?):

$$\{p_0,\ p_0 \to p_1,\ p_1\} \vdash p_2, \qquad \{p_0,\ p_0 \to p_1,\ \neg p_1\} \vdash p_2$$

Solution: The statement on the left is incorrect: take a valuation v with $v(p_0) = v(p_1) = 1$ and $v(p_2) = 0$. Then $v(p_0) = v(p_0 \to p_1) = v(p_1) = 1$, but $v(p_2) = 0$, so

$$\{p_0,\ p_0 \to p_1,\ p_1\} \not\models p_2,$$

hence by Soundness, $\{p_0,\ p_0 \to p_1,\ p_1\} \nvdash p_2$.

The statement on the right is correct: one can either give a derivation witnessing it (easy), or appeal to completeness, by which it is enough to check

$$\{p_0,\ p_0 \to p_1,\ \neg p_1\} \models p_2,$$

which follows by noting that there is no valuation v such that $v(p_0) = v(p_0 \to p_1) = v(\neg p_1) = 1$.

(2) Let $L = \{<,\ +,\ \cdot,\ 0,\ 1\}$ be the language whose symbols are the binary relation symbol $<$, the binary function symbols $+$ and $\cdot$, and the constant symbols 0 and 1. Let x, y, z be distinct variables. Let t be the L-term xy. (Here xy abbreviates $\cdot(x, y)$; likewise $x + z$ abbreviates $+(x, z)$, and so on.) Let ϕ be the L-formula

$$x \neq 0 \vee y = 1 \vee \exists y(y \neq 1 \wedge y < x + z)$$

(i) which variables occur free in ϕ, and which occur bound?
(ii) determine the formulas $\phi[t/x]$, $\phi[t/y]$, and $\phi[t/z]$.
(iii) Is t is free for x in ϕ? Same question for y and for z instead of x (same t).

Here is is enough just to give the answers.

Solution:

(i) x, y, z occur free in ϕ, and only y occurs bound in ϕ.
(ii) $\phi[t/x]$ is $xy \neq 0\ \vee\ y = 1\ \vee \exists y(y \neq 1\ \wedge\ y < xy + z)$
$\phi[t/y]$ is $x \neq 0\ \vee\ xy = 1\ \vee\ \exists y(y \neq 1\ \wedge\ y < x + z)$
$\phi[t/z]$ is $x \neq 0\ \vee\ y = 1\ \vee\ \exists y(y \neq 1\ \wedge\ y < x + xy)$
(iii) t is not free for x in ϕ; t is free for y in ϕ, and t is not free for z in ϕ.

(3) Let the language L have a unary predicate symbol P; let y, z be distinct variables. Determine which of (i), (ii), (iii), (iv) below are correct, and which are not:

(i) $\models P(y) \to \exists y P(y)$
(ii) $\models P(y) \to \forall y P(y)$
(iii) $\models (\forall y P(y)) \to P(z)$
(iv) $\models \forall y \exists z \big(z \neq y \wedge P(z)\big) \;\longleftrightarrow\; \exists y \exists z \big(y \neq z \wedge P(y) \wedge P(z)\big)$

Here is is enough just to give the answers.

Solution: $(1), (3), (4)$ are correct, (2) is not.

(4) Let P, Q be a binary relation symbols and x, y, z distinct variables. Indicate prenex transformations that convert the formula

$$\big(\forall x P(x,y)\big) \to \exists x \forall y Q(x,y)$$

into a semantically equivalent formula in prenex normal form.

Solution: First, change to $\neg\big(\forall x P(x,y)\big) \ \vee \ \exists x \forall y Q(x,y)$, and change that to $\big(\exists x \neg P(x,y)\big) \vee \exists x \forall y Q(x,y)$; transform that into $\exists x\big(\neg P(x,y) \vee \forall y Q(x,y)\big)$. Next we replace the bounded occurrences of y to v, where the variable v differs from x, y to get $\exists x\big(\neg P(x,y) \ \vee \ \forall v Q(x,v)\big)$, which we then transform into $\exists x \forall v\big(\neg P(x,y) \vee Q(x,v)\big)$. If we like we can transform that further to $\exists x \forall v\big(P(x,y) \to Q(x,v)\big)$.

(5) Let the language L have just the constant symbols 0 and 1 and the binary function symbol $\cdot$. Consider the L-structures

$$\langle \mathbb{N}; 0, 1, \cdot \rangle \qquad \langle \mathbb{Z}; 0, 1, \cdot \rangle \qquad \langle \mathbb{Q}; 0, 1, \cdot \rangle$$

with the usual interpretation of these symbols. Find for each of these three structures an L-sentence that is true in it and false in the other two.

Solution: the sentence $\sigma_1 := \forall x(xx = 1 \to x = 1)$ is true in $\langle \mathbb{N}; 0, 1, \cdot \rangle$ and false in the other two. The sentence $\sigma_3 := \forall x\big((x \neq 0 \to \exists y(xy = 1)\big)$ is true in $\langle \mathbb{Q}; 0, 1, \cdot \rangle$ and false in the other two, and so $\sigma_2 := \neg\sigma_1 \wedge \neg\sigma_3$ is true in $\langle \mathbb{Z}; 0, 1, \cdot \rangle$ and false in the other two.

(6) Let ϕ, ψ be L-formulas such that the variable x does not occur free in ϕ. Give derivations (with a label for each step indicating the rule being used) showing:

(i) $\forall x(\phi \to \psi) \vdash \phi \to \forall x\psi$
(ii) $\phi \to \forall x\psi \vdash \forall x(\phi \to \psi)$.