(1) How many words are there of length \( \leq 4 \) on the alphabet \( \{a, b, c, d, e\} \)?

**Solution:** There are \( 5^n \) words of length \( n \) on a five-letter alphabet, so the answer is
\[
1 + 5 + 5^2 + 5^3 + 5^4 = 1 + 5 + 25 + 125 + 625 = 781.
\]

(2) Find semantically equivalent simpler propositions for:
\[
p_0 \lor (p_1 \land p_0), \quad (p_1 \rightarrow p_0) \rightarrow p_1
\]

**Solution:** \( p_0 \lor (p_1 \land p_0) \) is semantically equivalent to \( p_0 \), and \( (p_1 \rightarrow p_0) \rightarrow p_1 \) is semantically equivalent to \( p_1 \).

(3) Determine disjunctive normal forms for:
\[
\neg (p_0 \leftrightarrow p_2), \quad (p_0 \lor p_1) \rightarrow p_2
\]

**Solution:** \( (p_0 \land \neg p_2) \lor (\neg p_0 \land p_2) \) is a disjunctive normal form for \( \neg (p_0 \leftrightarrow p_2) \), and \( (\neg p_0 \land \neg p_1) \lor p_2 \) is a disjunctive normal form for \( (p_0 \lor p_1) \rightarrow p_2 \).

(4) Enumerate the set of all valuations \( v \) such that for all \( n \),
\[
v(p_n \rightarrow p_{n+1}) = 1
\]

(This means in particular showing that this set is countable.)

**Solution:** One \( v \) (call it \( v_{-1} \)) that satisfies the condition is given by \( v(p_n) = 0 \) for all \( n \). For any other \( v \) that satisfies the condition there is a least \( m \) with \( v(p_m) = 1 \), hence \( v(p_{m+1}) = 1 \), so \( v(p_{m+2}) = 1 \), and so on, that is, \( v(p_n) = 1 \) for all \( n \geq m \). Given any \( m \), the valuation \( v_m \) defined by \( v_m(p_n) = 0 \) if \( n < m \) and \( v_m(p_n) = 1 \) if \( n \geq m \) is indeed a valuation satisfying the condition. So the set of valuations satisfying the condition is \( \{v_{-1}, v_0, v_1, v_2, \ldots \} \).

For solutions of the last two problems, see back of sheet.

(5) Show: \( \vdash [(\phi \rightarrow \psi) \land (\phi \rightarrow \neg \psi)] \rightarrow \neg \phi \)
by giving a derivation.

(6) Show: \( \vdash [(\phi \rightarrow \theta) \land (\psi \rightarrow \theta)] \rightarrow [(\phi \lor \psi) \rightarrow \theta] \)
by giving a derivation that uses the \( \lor \)-elimination rule at a key step, and without rewriting \( \phi \lor \psi \) as \( \neg (\neg \phi \land \neg \psi) \).