Midterm Math 414 (Logic), February 18, 2017

(1) How many words are there of length ≤ 4 on the alphabet $\{a, b, c, d, e\}$?
(Hint: more than 100, less than 1,000)

Solution: There are 5^n words of length n on a five-letter alphabet, so the answer is

$$1 + 5 + 5^2 + 5^3 + 5^4 \;=\; 1 + 5 + 25 + 125 + 625 \;=\; 781.$$

(2) Find semantically equivalent simpler propositions for:

$$p_0 \vee (p_1 \wedge p_0), \qquad (p_1 \to p_0) \to p_1$$

Solution: $p_0 \vee (p_1 \wedge p_0)$ is semantically equivalent to p_0, and $(p_1 \to p_0) \to p_1$ is semantically equivalent to p_1.

(3) Determine disjunctive normal forms for:

$$\neg(p_0 \leftrightarrow p_2), \qquad (p_0 \vee p_1) \to p_2$$

Solution: $(p_0 \wedge \neg p_2) \vee (\neg p_0 \wedge p_2)$ is a disjunctive normal form for $\neg(p_0 \leftrightarrow p_2)$, and $(\neg p_0 \wedge \neg p_1) \vee p_2$ is a disjunctive normal form for $(p_0 \vee p_1) \to p_2$.

(4) Enumerate the set of all valuations v such that for all n,

$$v(p_n \to p_{n+1}) = 1$$

(This means in particular showing that this set is countable.)

Solution: One v (call it v_{-1}) that satisfies the condition is given by $v(p_n) = 0$ for all n. For any other v that satisfies the condition there is a least m with $v(p_m) = 1$, hence $v(p_{m+1}) = 1$, so $v(p_{m+2}) = 1$, and so on, that is, $v(p_n) = 1$ for all $n \geq m$. Given any m, the valuation v_m defined by $v_m(p_n) = 0$ if $n < m$ and $v_m(p_n) = 1$ if $n \geq m$ is indeed a valuation satisfying the condition. So the set of valuations satisfying the condition is $\{v_{-1}, v_0, v_1, v_2, \cdots\}$.

For solutions of the last two problems, see back of sheet.

(5) Show: $\quad \vdash [(\phi \to \psi) \wedge (\phi \to \neg\psi)] \to \neg\phi$
by giving a derivation.

(6) Show: $\quad \vdash [(\phi \to \theta) \wedge (\psi \to \theta)] \to [(\phi \vee \psi) \to \theta]$
by giving a derivation that uses the $\vee$-elimination rule at a key step, and without rewriting $\phi \vee \psi$ as $\neg(\neg\phi \wedge \neg\psi)$.