Distributions and Topological Vector Spaces

Consider a topological space $X$ with topology $\mathcal{T}$. We write $(X, \mathcal{T})$.

**Definition:** A family $\mathcal{B} \subset \mathcal{T}$ is called a base if any $A \in \mathcal{T}$ is of the form $A = \bigcup_{x} B_{x}$ for some $\{B_{x}\} \subset \mathcal{B}$.

**Definition:** Let $(X, \mathcal{T})$ a topological space and $\mathcal{F}$ a family of functions from a set $S$ to $(X, \mathcal{T})$. The $\mathcal{F}$-weak topology on $S$ is the weakest topology $(\mathcal{T}_1 < \mathcal{T}_2 \iff \mathcal{T}_1$ is weaker than $\mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$) for which all functions $f \in \mathcal{F}$ are continuous.

To construct the weak topology take $\mathcal{N}_{f^{-1}}(U)$ where $f \in \mathcal{F}$ and $U \in \mathcal{T}$. These sets form a base for the $\mathcal{F}$-weak topology.

**Definition:** A family of subsets $\mathcal{N} \subset X$ is called a neighborhood base at $x$ if each $\mathcal{N} \in \mathcal{N}$ is a neighborhood at $x$ $(\exists U$ open with $x \in U \subset N)$ and given any neighborhood $M \ni x$, there is $N \subset M$ with $N \in \mathcal{N}$.

- If each point $x \in X$ has a countable neighborhood base, then $X$ is first countable.
- If $X$ has a countable base, then $X$ is called second countable.

**Remark:** Since if $\mathcal{B}$ is a base for $\mathcal{T} \Rightarrow \{N \in \mathcal{B} : x \in N\}$ is a neighborhood base at $x$, every second countable topological space is first countable.

For metric spaces we have:
1. Every metric space is first countable (closed balls with rational radius around $x$ is a neighborhood base at $x$).
2. A metric space is second countable iff it is separable.
3. Any second countable topological space is separable.
In spaces that are first countable we can construct closures of sets by using only sequences. For general spaces, sequences are not enough.

Prop 4.6 pg 110 in Folland: If $X$ is first countable and $A \subseteq X$ then $x \in \overline{A}$ if and only if there exists a sequence $(x_j) \subseteq A$ such that $x_j \to x$. (For every neighborhood $U$ of $x$, there exists $J \in \mathbb{N}$ such that $x_j \in U$ for all $j \geq J$.)

Definition: $X$ is Hausdorff if for any two distinct points $x, y \in X$, there exist disjoint open sets $U, V$ with $x \in U$ and $y \in V$. (Hausdorff leads to the uniqueness of the limit in topological spaces that satisfy the axiom of first countability.)

Example: $S = [0, 1]$. Define $\mathcal{T}$ by the requirement that the nonempty open sets be those subsets of $S$ whose complements contain at most a countable infinity of points. $S$ is obviously uncountable. Let $A = [0, 1]$. Then $\overline{A} = S$. Let $(x_n)$ be any sequence of points in $[0, 1]$. The complement of the set $\{x_0, x_1, \ldots, x_n, \ldots\}$ is open and contains $1$. Thus, $(x_n)$ cannot converge to $1 \in \overline{A}$.

To remedy this pathology, we introduce the nets. Note that for infinite dimensional Banach spaces, the weak topology does not arise from a metric (it is not metrizable). This is the reason we cannot avoid talking about topological spaces in analysis.

Definition: A directed set is a set $A$ together with a relation $\leq$ such that:

i) $\not\exists x$ for all $x \in A$.
ii) $x \leq y$ and $y \leq z \Rightarrow x \leq z$.
iii) For any $x, y \in A$, $x \leq y$ and $y \leq x$.

$\mathbb{N}$ is a directed set with $j \leq k$ if and only if $j \leq k$.

A net in a set $X$ is a mapping from a directed set $A$ to $X$. If $A = \mathbb{N}$, we have sequences.
\( (x_\alpha)_{\alpha \in A} \) converges to \( x \in T \) if for any neighborhood \( N \) of \( x \) there is \( \alpha \in A \) so that \( x_\alpha \in N \) if \( \alpha > \beta \).

We this definition we have

**Proposition (Fellow 120-121)** Let \( S \) be a set in a topological space \( T \). Then \( x \in \overline{S} \) iff \( x \) is a net \( (x_\alpha)_{\alpha \in A} \) with \( x_\alpha \in S \) \( \forall \alpha \) and \( x_\alpha \to x \).

ii) \( f : S \to T \) is continuous iff for every \( (x_\alpha)_{\alpha \in A} \) with \( x_\alpha \to x \) we have

\[ \{f(x_\alpha)\}_{\alpha \in A} \to f(x) \text{ in } T. \]

**Definition** A seminorm on a vector space \( V \) is a map \( p : V \to [0, \infty) \) obeying

\[ p(x+y) \leq p(x) + p(y) \]

\[ p(ax) = |a| p(x) \quad \forall x \in V, \quad a \in \mathbb{C}, \quad \text{and \ it \ separate \ points \ if \ } p \]

A family of seminorms \( \{p_\alpha\}_{\alpha \in A} \) is said to separate points if \( p_\alpha(x) = 0 \) for all \( x \) implies \( x = 0 \).

**Definition** A locally convex space is a vector space over \( \mathbb{C} \) or \( \mathbb{R} \) with a family of seminorms \( \{p_\alpha\}_{\alpha \in A} \) that separate points. The natural topology on the locally convex space is the weakest topology in which all \( p_\alpha \) are continuous and in which the operation of addition and multiplication are continuous \( (x,y) \mapsto x+y \) \( x \in V \to x \)

\[ (\lambda, x) \mapsto \lambda x \quad \forall x \in V \to x. \]

**Remark** The space is called locally convex if there is a base for the topology consisting of convex sets. But this base comes from the convex sets

\[ U_{x, \varepsilon} = \{ y \in X : p_\alpha(x-y) < \varepsilon \} \quad [x \in X, \alpha \in A, \varepsilon \in (0, \infty)] \]

in view of the following

**Proposition**
Follow (pg 159) Consider the topology generated by the sets $U_{\epsilon \in E}$. Then

1. $\bigcap_{X \in E}$ finite intersections of $U_{\epsilon \in E}$ form a neighborhood base at $x$.

The neighborhood base at $0$ is given by $U_{\epsilon, n_{i}, x} = \{ x : p_{x_{i}}(x) < \epsilon, i = 1, 2, \ldots, n \}$

2. $x \in X \iff \bigcap_{\epsilon} p_{x}(x_{\epsilon} - x) \to 0$ for all $\epsilon \in E$.

Since $\{ p_{x} \}_{x}$ separates points the natural topology is Hausdorff.

We can actually characterize this topology on a locally convex space.

Definition: A set $C \subseteq V$, a vector space, is called convex if $x$ and $y \in V$, $0 \leq t \leq 1$ implies $tx + (1 - t)y \in C$. $C$ is called balanced if $x \in C$ and $|x| = 1$ implies $-x \in C$. $C$ is called absorbing if $V = \cup tC$ that is $V = \bigcup_{t>0} tC$ for some $t > 0$.

Proposition: If $p_{x_{1}}, p_{x_{2}}, \ldots, p_{x_{n}}$ are seminorms on a vector space $V$ then

$\{ x \in p_{x_{1}}(x) \leq \epsilon, \ldots, p_{x_{n}}(x) \leq \epsilon \}$ is balanced, convex and absorbing set.

Thus a locally convex space that is generated by the seminorms $p_{x}$ has a neighborhood base at $0$ of balanced, convex and absorbing sets.

Conversely, if $V$ has a neighborhood base at $0$ of balanced, convex and absorbing sets then $V$ is a locally convex topological space. Of course we assume that $V$ is a topological space with a Hausdorff topology in which addition and multiplication are separately continuous.

For the proof look at Reed and Simon Volume I.
The following is the analogue of continuity properties and their implications between normed spaces.

**Theorem (Folland pg. 159)** Suppose \( X \) and \( Y \) are vector spaces with topologies defined respectively by the families \( \{p_a\}_{a \in A} \) and \( \{q_b\}_{b \in B} \) of seminorms and \( T : X \to Y \) is a linear map. \( T \) is continuous if for each \( b \in B \) there exists \( a_1, a_2, \ldots, a_k \in A \) and \( c > 0 \) such that

\[
q_b(Tx) \leq c \sum_{j=1}^{k} p_{a_j}(x)
\]

Thus a linear functional on \( X \) is continuous if \( |f(x)| \leq c \sum_{j=1}^{k} p_{a_j}(x) \) for some \( c > 0 \) and \( a_1, a_2, \ldots, a_k \in A \). Since the finite sum of seminorms is again a seminorm, the Hahn–Banach theorem guarantees the existence of lots of continuous linear functionals on \( X \), enough to separate points if \( X \) is Hausdorff.

**Question:** Which locally convex spaces lead to complete metric spaces? The following is fundamental (Folland pg. 159).

Let \( X \) be a vector space with the topology defined by the family \( \{p_a\}_{a \in A} \) of seminorms that separate points (thus \( X \) is Hausdorff). Then if \( A \) is countable, \( X \) is metrizable (the topology that is generated by the countable family of seminorms is the same as the topology that is generated by the metric defined on \( X \times X \))

\[
d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left[ \frac{p_n(x - y)}{1 + p_n(x - y)} \right]
\]

**Definition:** A complete, metrizable, locally convex topological space is called a Fréchet.

An example of a Fréchet space is the vector space \( F(\mathbb{R}) \) whose topology is...
is generated by the seminorms \( \| f \|_{\alpha, N} = \sup_{x \in \mathbb{R}^n} (\alpha \cdot |x|) |D^\alpha f(x)| \) for \( \alpha \) a multi-index and \( N \in \mathbb{N} \).

\( S(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : \| f \|_{\alpha, N} < \infty \text{ for all } \alpha, N \} \)

The fact that \( \| \cdot \|_{\alpha, N} \) is a seminorm is easy. Since there countable many of them \( S \) is metrizable. Completeness follows by Prop 8.1 in Folland.

Now let's construct a topology on \( C^\infty(\mathbb{R}^n) \) (\( C^\infty \) functions with compact support).

\[ \text{supp} f = f^{-1}([0, \infty)) = \{ x \in \mathbb{R}^n : |f(x)| > 0 \} = \{ x \in \mathbb{R}^n : f(x) \neq 0 \} \]

In general consider \( f \in C^\infty(\mathbb{R}^n) \) an open set. We say that \( f \in C^\infty(\mathbb{R}^n) \) for every multi-index \( \alpha \).

If \( K \) is compact then \( D_K(\mathbb{R}^n) \) is the set of \( f \in C^\infty(\mathbb{R}^n) \) whose support lies in \( K \).

If \( K \subset \mathbb{R}^n \) then \( D_K \subset C^\infty(\mathbb{R}^n) \).

We can define the seminorms \( P_N, \text{ } N = 1, 2, \ldots \) on \( C^\infty(\mathbb{R}^n) \) by

\[ (1) \quad P_N(f) = \max \left\{ |D^\alpha f(x)| : x \in K_N, |\alpha| \leq N \right\} \]

where \( K_N \) is a compact set such that each \( K_N \) lies in the interior of \( K_{N+1} \) and \( \mathbb{R}^n = \bigcup K_N \).

These countable family of seminorms define a metrizable locally convex topology on \( C^\infty(\mathbb{R}^n) \). The local base is given by

\[ V_N = \left\{ f \in C^\infty(\mathbb{R}^n) : \frac{P_N(f)}{N} < 1 \right\} \quad N = 1, 2, \ldots \]

The space is complete and the \( C^\infty(\mathbb{R}^n) \) is Frechet.

Since for each \( x \in \mathbb{R}^n \) the functional \( y(f) = f(x) \) is continuous in this topology, the null spaces of these functionals are continuous.
$D_k$ is the intersection of the null space of all functionals as $x$ ranges over $\mathbb{R}$.

The complement of $K$ and the $D_k(\mathbb{R})$ is a closed subset of $C^0(\mathbb{R})$ and there is a Frechet for every $K$.

Now for $D(\mathbb{R}) = C^0(\mathbb{R})$ we can introduce the norms $\|\psi\|_N = \max \{ |D^k\psi(x)| : x \in \mathbb{R}, 1 \leq k \leq N \}$ for yet $D(\mathbb{R})$. This defines a locally convex metrizable space as before but the following example shows that $D(\mathbb{R})$ is not complete.

Take any $\psi \in C^0(\mathbb{R})$ such that $\text{supp}(\psi) = [0,1]$ and $\psi(x) > 0$ for $x \in [0,1)$. For $n \geq 1$ define

$$\psi_n(x) = \sum_{j=1}^{n} \frac{1}{j} \psi(x-j) \in C^0(\mathbb{R})$$

with $\text{supp} \psi = [1+n+1]$. Define $\psi \in C^0(\mathbb{R}) \setminus C^0(\mathbb{R})$ : $\psi(x) = \sum_{j=1}^{\infty} \frac{1}{j} \psi(x-j)$. It is easy to see that

$$\|\psi_n - \psi\|_N \to 0 \text{ as } n \to \infty$$

but the limit is not in $D(\mathbb{R})$.

We can now define another locally convex topology on $D(\mathbb{R})$ in which Cauchy sequences converge and the $D(\mathbb{R})$ is complete. The fact that $D(\mathbb{R})$ in this topology is not metrizable is only a minor inconvenience.

The result of this construction leads to the pleasant property that every differential operator $D^\alpha$ is a continuous mapping of $D(\mathbb{R})$ into $D(\mathbb{R})$.

Before going into the details let's stress out one more time the usefulness of the construction of topological spaces whose topology is not given by a norm.

Recall the operator $\frac{d}{dx}$. It is impossible to define norms on most infinite-dimensional...
function spaces in which $\frac{d}{dx}$ becomes a bounded operator. For example, if we take $C^\infty([0,1])$ and consider $f_\lambda(x) = e^{\lambda x}$, then $\frac{d}{dx}f_\lambda = \lambda f_\lambda$ and thus $\|\frac{d}{dx}\| \geq 1|\lambda|$ for all $\lambda$, no matter what norm is used. In view of this difficulty three courses of action are available:

a) Consider $\frac{d}{dx}$ as an unbounded operator from $X$ to $Y$ where $Y$ is Banach and $X$ is dense in $Y$. We have seen in class this construction.

b) Bound $\frac{d}{dx}$ within two different spaces such as $X = C^k([0,1])$ and $Y = C([0,1])$.

c) Consider differentiation as a continuous operator on a locally convex space $X$ whose topology is not given by a norm. For example, if we consider the seminorms

$$P_k(f) = \max_{0 \leq x \leq 1} |f^{(k)}(x)|, \quad k = 1, 2, \ldots$$

then we know that $C^\infty([0,1])$ becomes a Fréchet space and $\frac{d}{dx}$ is continuous by the Theorem on page 3 since $P_k\left(\frac{d}{dx}\right) = P_{k+1}(f)$.

**Theorem 4 (Reed and Simon, Chapter 5, Vol I)**

Let $X$ a complex (or real) vector space. Let $X_n$ be a family of subspaces with $X_n \subseteq X_{n+1}$, $X = \bigcup_{n=1}^{\infty} X_n$. Suppose that each $X_n$ has a locally convex topology so that the restriction of the topology of $X_{n+1}$ to $X_n$ is the given topology on $X_n$. Let $\mathcal{U}$ a collection balanced, absorbing, convex sets $0$ in $X$ for which $0 \cap X_n$ is open in $X_n$ for each $n$. Then

a) $\mathcal{U}$ is a neighborhood base at $0$ for a locally convex topology.
b) The topology generated by \( U \) is the strongest topology on \( X \) so that the injections \( X_n \rightarrow X \) are continuous.

c) The restriction of the topology on \( X \) to each \( X_n \) is the given topology on \( X_n \).

d) If each \( X_n \) is complete then so is \( X \).

**Definition:** \( X \) is called the strict inductive limit of \( X_n \).

An easy byproduct of Theorem 1 is:

**Theorem 2:** Let \( X \) be the strict inductive limit of the locally convex spaces \( \{X_n\}_{n=1}^\infty \). Then a linear map from \( X \) to a locally convex space \( Y \) is continuous iff each of the restrictions \( T|_{X_n} \) is continuous.

Now let \( O = \bigcap_{i=1}^\infty K_i \) with \( K_i \) compact and \( K_i \subset K_i^\circ \) (the interior of \( K_i \)).

We have seen that \( C^\infty_0(K_i) \) has the Fréchet topology generated by the seminorms

\[
p_n(f) = \max_{1 \leq N} \{ \|D^N f\| : f \in K_N \}, \quad n \leq N\}
\]

Then \( C^\infty_0(O) = D(O) = \bigcup_{i=1}^\infty C^\infty_0(K_i) \)

The topology is independent of the choice of \( K_i \).

**\( D(O) \) is called an LF-space** if it is not metrizable.

**Let \( 0 < M < N \).** Then \( D_M \) is a closed subset of \( D_N \) since the property of vanishing off \( K_M \) is preserved under sup-norm limits. In addition any open ball of radius \( \varepsilon > 0 \) around any function in \( D_M \) contains many functions with non-zero values off \( K_M \). Thus \( D_M \) is nowhere dense in \( D_N \). This is each \( D_K \) is nowhere dense in \( D(O) \) while 

\[
D(O) = \bigcup_{i=1}^\infty D_K_i
\]
By Baire's category theorem, the topology on $D(\mathbb{R})$ cannot be given by a complete metric.

Theorem 3: Suppose $X = \bigcup X_n$ has a strict inductive topology and that each $X_n$ is a closed proper subspace of $X$. Then a sequence $f_m \in X$ converges to $f \in X$ iff all $f_m$ are in some $X_n$ and $f_m \to f$ in the topology of that $X_n$.

The proof of this theorem can be found in Reed and Simon, Vol I, Theorem IV.17. In particular, a sequence $f_m \in D'(\mathbb{R}^n)$ converges to $f \in D'(\mathbb{R}^n)$ iff all the $f_m$ and $f$ have support inside some fixed compact $K$ and $D^\alpha f_m$ converges uniformly to $D^\alpha f$ for each multi-index $\alpha$. Here of course convergence means convergence in the topology that arises in the relevant seminorms $(x_\delta \to x$ iff $\|x_\delta - x\|_x \to 0$ for $x(x^\delta)\).

Definition: A generalized function (or distribution) is a continuous linear functional on $D(\mathbb{R})$. This space is denoted by $D'(\mathbb{R})$.

The previous discussion proves the basis corollary:

A linear functional $T$ on $D'(\mathbb{R}^n)$ is continuous iff for each compact $K \subseteq \mathbb{R}^n$ there is $C$ and $\gamma \in \mathbb{N}^n$ so that

$$|T(\psi)| \leq C \sum_{|\gamma| \leq \gamma} \|D^\gamma \psi\|_0$$

for all $\psi \in C^\infty_0 (K)$.

Notice that the seminorm on $C$ is just the absolute value.

Some further remarks that connect the above discussion with certain ideas already discussed in class:

1. We have seen that $f \in \mathcal{S}(\mathbb{R}^n)$ iff $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$. This will help us define the Fourier transform for every continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$. 

But not all distributions have Fourier transforms. The difficulty arises from the observation that if $f \in C_0^\infty(\mathbb{R}^n)$ and $\hat{f} \neq 0$ then $\hat{f} \notin C_0^\infty(\mathbb{R}^n)$.

If $f \in C_0^\infty(\mathbb{R}^n)$ we can define $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x) \, dx$. By differentiating under the integral sign one sees that $\hat{f}$ extends to a holomorphic in all of $C_0^\infty(\text{entire})$.

But entire functions do not have compact support.