Criteria for self-adjointness

Recall that an unbounded operator $T: D \subset H \rightarrow H$ where $H$ is a Hilbert space and $\overline{D} = H$ is closed if the graph of $T$

\[ G(T) = \{(x, Tx) : x \in D\} \subset H \times H \] is closed in $H \times H$. Or $T$ is closed if for any $x_n \in D$ with $x_n \rightarrow x$ and $Tx_n \rightarrow y$ we have $x \in D$ and $y = Tx$.

Definition: A linear operator is **closed** if it has an extension which is closed. The smallest closed extension of $T$ is called the **closure** of $T$ and is denoted by $\overline{T}$. $T$ is closed if $\overline{T} = T$.

Remark: By continuity a bounded linear operator is closed if $D$ is closed in $H$.

Let $\ker T = \{x \in D : Tx = 0\}$, $R_T = \{y \in H : Tx = y \text{ for some } x \in D\}$

By the closed graph theorem ($G(T)$ closed $\Rightarrow T$ bounded) we
know that if $\ker T = \{0\}$, $\text{R}_T = H$ and $G(T)$ is closed then $\overline{T^{-1}} : H \rightarrow H$ is bounded.

3. It can be proved that a linear operator is closable iff the closure of its graph, $\overline{G(T)}$, is a graph and in this case $G(\overline{T}) = \overline{G(T)}$.

3. We saw in class that $T^*$ is always closed. It can be proved that $T$ is closable iff $T^*$ is densely defined and $\overline{T} = T^{**}$.

4. If $z \in \text{R}_T^{\perp} \Rightarrow \langle z, Tx \rangle = \langle T^* z, x \rangle$. Since $D$ is dense,

and $\langle T^* z, x \rangle = 0$ for all $x \in D \Rightarrow T^* z = 0 \Rightarrow z \in \ker T^*$.

Reversing the steps we get $\ker T^* = \text{R}_T^{\perp} \quad (i)$

If in addition $T$ is closed (thus $T^{**} = T$) $(i)$ gives that

$\text{R}_{T^*}^{\perp} = \ker T^{**} = \ker T \quad (ii)$

For a symmetric operator $\text{R}(T) \in \mathbb{R}$. This is because by symmetry $\langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\langle T x, x \rangle}$ for all
The \( \langle Tx, x \rangle \in \mathbb{R} \) and if \( \lambda \in \mathcal{D}(T) \Rightarrow Tx = \lambda x \) then \( \lambda \|x\|^2 \in \mathbb{R} \Rightarrow \lambda \in \mathbb{R} \). The stronger conclusion \( \mathcal{G}(T) \subset \mathbb{R} \) required \( T \) to be self-adjoint.

**Lemma:** If \( \text{Im} \lambda \neq 0 \) then \( \mathbb{R}_{T-\lambda} \) is closed.

**Proof:** Let \( x = \alpha + i\beta \) with \( \beta \neq 0 \). For any \( x \in \mathcal{D} \) we have
\[
\| (T-\lambda) x \|^2 = \| (T-\alpha) x \|^2 + 2 \Re \lambda \langle (T-\alpha) x, \beta x \rangle + \beta^2 \| x \|^2
\]
But \( \langle (T-\alpha) x, \beta x \rangle = \langle Tx, \beta x \rangle - \alpha \beta \| x \|^2 = \beta \langle Tx, x \rangle - \alpha \beta \| x \|^2 \in \mathbb{R} \) since \( \mathcal{G}(T) \subset \mathbb{R} \). Thus \( \Re \lambda \langle (T-\alpha) x, \beta x \rangle = 0 \) and
\[
\| (T-\lambda) x \|^2 = \| (T-\alpha) x \|^2 + \beta^2 \| x \|^2 \geq \beta^2 \| x \|^2
\] where \( \beta > 0 \). Now let \( y \in \mathbb{R}_{T-\lambda} \Rightarrow \exists x \in \mathcal{D} \ni (T-\lambda)x \to y \)

But \( \| x_n \|^2 \leq \frac{1}{\beta^2} \| (T-\lambda)x_n \|^2 \) and \( \{ x_n \} \) is Cauchy and thus converges to \( x \in \mathcal{H} \). Since \( T-\lambda I \) is closed we have that \( x \in \mathcal{D} \) and \( (T-\lambda)x = y \Rightarrow y \in \mathbb{R}_{T-\lambda} \).

**Theorem:** Suppose \( T : \mathcal{D} \to \mathcal{X} \) is a closed symmetric operator on \( \mathcal{H} \). The
following are equivalent

(i) $T$ is self-adjoint

(ii) $\ker (T^* - \lambda I) = \{0\}$ for all $\lambda \in \mathbb{C}$ with $\text{Im} \lambda \neq 0$

(iii) $\mathcal{G}(T) \subseteq \mathbb{R}$

Proof: (i) $\Rightarrow$ (ii) If $\lambda \in \mathbb{C}$ with $\text{Im} \lambda \neq 0$ and $T = T^*$ then

If $x \in \ker (T^* - \lambda I) \Rightarrow T^* x = \lambda x \quad \forall x \in \mathcal{D}$

But then

$\bar{\lambda} \langle x, x \rangle = \langle x, \lambda x \rangle = \langle x, T x \rangle = \langle T^* x, x \rangle$

$= \lambda \langle x, x \rangle \Rightarrow (\lambda - \bar{\lambda}) \| x \|^2 = 0 \Rightarrow x = 0$ since $\lambda \neq \bar{\lambda}$

(ii) $\Rightarrow$ (iii) Since $T$ is symmetric and $\lambda \in \mathbb{C}$ with $\text{Im} \lambda \neq 0$ then $\ker (T - \lambda I) = \{0\}$. By the hypothesis $\ker (T^* - \lambda) = \{0\}$ and by remark (4) $\mathcal{R}^\perp_{T - \lambda} = \{0\}$ and thus $\mathcal{R}_{T - \lambda} = \mathbb{H}$

Since $T - \lambda I$ is closed by remark (2) $(T - \lambda I)^{-1} : \mathbb{H} \rightarrow \mathbb{H}$ is bounded and $\lambda \in \rho (T)$. We have shown that $G(T) \subseteq \mathbb{R}$

(iii) $\Rightarrow$ (i) Since $Tc \subseteq T^*$ we want to show that if $y \in \mathcal{D}$
dom \( (T^* + i) \) \( \Rightarrow \) \( y \in D \). Since \( g(T) \in \mathbb{R} \) the operator \( T + i \mathbb{I} : H \rightarrow H \) is boundedly invertible and thus for \( \forall y \in H \) \( \exists x \in H \) \( (T^* + i)x = (T^* + i)y \)

Since \( T + i \mathbb{I} \subseteq T^* + i \mathbb{I} \Rightarrow (T^* + i)x = (T + i)x \) for \( x \in D \).

Thus \( (T^* + i)(x - y) = 0 \) \( \Rightarrow \) \( y = x \in D \) since \( \ker(T^* - \lambda) = \{0\} \) for all \( \lambda \in \mathbb{C} \) with \( \lambda \neq 0 \).