Trace zero functions in $W^{1,p}(U)$ / Characterization of $W_0^{1,p}(U)$
when $U$ is bounded with smooth boundary.

Assume $U$ is bounded with $C^1$ boundary. Assume further that $u \in W^{1,p}(U)$.
Then $u \in W_0^{1,p}(U)$ iff $Tu = 0$ on $\partial U$.

Proof: $\Leftarrow$ Let $u \in W_0^{1,p}(U)$ then $\exists \, \psi \in C_0^\infty(U)$ such that
$$u_m \rightarrow u \quad \text{in} \quad W^{1,p}(U)$$

But since $u_m \in C_0^\infty(U)$ then $Tu_m = u_m |_{\partial U} = 0$ and therefore
$$||Tu_m - Tu||_{L^p(\partial U)} = ||Tu_m - Tu||_{L^p(\partial U)} \leq C \, ||u_m - u||_{W^{1,p}(U)} \rightarrow 0$$
as $m \rightarrow \infty$ and thus $Tu = 0$ on $\partial U$.

$\Rightarrow$ Assume $Tu = 0$ on $\partial U$.

Remark

Assume that $u \in W^{1,p}(U)$ and we prove that $u \in W_0^{1,p}(U)$ when $u$ has compact support. Then pick for $V \subset U$, $J \in C_0^\infty(U)$ with

$J = 1$ on $\sqrt{2}$ and set $J_n = J(\frac{x}{n})$. Then $J_n u$ is compactly supported
in $U$ and by the same proof it follows that $J_n u \in W_0^{1,p}(U)$.

But $J_n u \rightarrow u$ in $W^{1,p}(U)$. Thus $J_n u \in W_0^{1,p}(U) \Rightarrow \exists \, \{u_m\}$
Thus, we can assume that $u$ has compact support in $\mathbb{R}^n$. Using again partitions of unity and flattening out $\mathbb{R}_+$ as usual, we may assume that

$$u \in W^{1,p}(\mathbb{R}^n_+), \quad u \text{ has compact support in } \mathbb{R}^n_+$$

We want to prove that $u \in W_0^{1,p}(\mathbb{R}^n_+)$.

But there exists $\{u_m\} \in C^1(\overline{\mathbb{R}^n_+})$ such that

1. $u_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$
2. and $\lim_{m \to \infty} u_m = u$ in $L^p(\mathbb{R}^{n-1})$ by assumption

Let $x_n = 0$, $x \in \mathbb{R}^{n-1}$ the

$$|u_m(x', x_n)| \leq |u_m(x', 0)| + \int_0^{x_n} \frac{2}{\partial x_n} u_m(x', t) \, dt$$

$$|u_m(x', x_n)|^p \leq C \left[ |u_m(x', 0)|^p + \int_0^{x_n} \frac{2}{\partial x_n} u_m(x', t) \, dt \right]^p$$

But

$$\int_0^{x_n} \frac{2}{\partial x_n} u_m(x', t) \, dt = \int_{\mathbb{R}^{n-1}} \frac{2}{\partial x_n} u_m(x', t) \, dt$$

$$\leq \left( \int_{\mathbb{R}^{n-1}} \frac{2}{\partial x_n} u_m(x', t) \, dt \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^{n-1}} \left| \frac{2}{\partial x_n} u_m(x', t) \right|^p \, dt \right)^\frac{1}{p}$$

$$= x_n^{1/q} \left( \int_{\mathbb{R}^{n-1}} \frac{2}{\partial x_n} u_m(x', t) \, dt \right)^{1/p}$$

Look at the proof of the extension operator on page 16, Book I, especially the set $Q^+ \to Q \to U; i \to U$ recalling that the proof...
The second term of the RHS of (3) is
\[ \leq \chi_\eta \int_0^\eta \left| \frac{\partial u_m}{\partial x} (x',t) \right|^p dt \]
\[ = \chi_\eta \int_0^{\eta} \left| \frac{\partial u_m}{\partial x} (x',t) \right|^p dt \quad \text{and if we integrate over } \mathbb{R}^{n-1}, \]
\[ \int_\mathbb{R}^{n-1} \left| \nu_m (x', x_n) \right|^p dx' \leq C \left( \left( \int_{\mathbb{R}^{n-1}} \left| u_m (x', 0) \right|^p dx' \right)^{p-1} \right)^{1/p} \int_\mathbb{R}^{n-1} \left| \frac{\partial u_m}{\partial x} (x', t) \right|^p dx' \]
\[ \leq C \chi_\eta \int_0^{\eta} \int_{\mathbb{R}^{n-1}} \left| \frac{\partial u_m}{\partial x} (x', t) \right|^p dx' dt \quad \text{(4)} \]

Now let \( f \in C^\infty (\mathbb{R}) \)

\[ \text{a bump function} \]

and write \( f_m (x) = f (mx) \)

\[ W_m = u(x) (1 - f_m) \]

\[ \frac{\partial W_m}{\partial x} = \frac{\partial u}{\partial x} (1 - f_m) - \frac{\partial u}{\partial x} f_m ' \]

\[ \frac{\partial W_m}{\partial x} = \frac{\partial u}{\partial x} (1 - f_m) \]

\[ \frac{\partial u}{\partial x} = \left( \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n} \right) \]

\[ \frac{\partial W_m}{\partial x} - \frac{\partial u}{\partial x} = \left( \frac{\partial u}{\partial x_1} f_m' - \frac{\partial u}{\partial x} f_m' \right) \quad \text{and the} \]

\[ \left| \frac{\partial W_m}{\partial x} - \frac{\partial u}{\partial x} \right|^p \leq C \left( \int_{\mathbb{R}^{n-1}} \left| \frac{\partial u}{\partial x} \right|^p dx' + m \int_{\mathbb{R}^{n-1}} \left| f_m' \right|^p dx' \right) \]

goes through with \( Q = \mathbb{R}^n \) and \( \Omega = \mathbb{R}_+^n \). The appearance of the reaction

on page 1
\[ \text{and that } \int \frac{\partial w_{m}}{\partial x} - \frac{\partial u}{\partial x} \right|^{p} \, dx \leq C \int |J_{m}|^{\frac{p}{2}} \left| \frac{\partial u}{\partial x} \right|^{p} \, dx \\
+ C m^{p} \int_{0}^{T} \int_{R^{n-1}} |u|^{p} \, dx \, dt \] \\
\text{since } \int\frac{f'}{t} = 0 \text{ for } t > \frac{2}{m}. \\
= A + B. \quad A \leq C \int_{0}^{2/m} \left| \frac{\partial u}{\partial x} \right|^{p} \, dx \to 0 \quad \text{as } m \to \infty. \\

\text{since } u \in W^{1,p}(R^{n})

\begin{align*}
B &\leq C m^{p} \left( \int_{0}^{2/m} \int_{0}^{T} |u|^{p} \, dt \, dx \right) \left( \int_{0}^{2/m} \int_{R^{n-1}} \frac{\partial u}{\partial x} \, dx \, dx' \, dx_{n} \right) \\
&\leq C \int_{0}^{2/m} \int_{R^{n-1}} \frac{\partial u}{\partial x} \, dx \, dx_{n} \to 0 \quad \text{as } m \to \infty. \\

\text{Thus, } \frac{\partial u}{\partial x} \to \frac{\partial u}{\partial x} \text{ in } L^{p}(R^{n}) \text{ and since } W_{m} \Rightarrow u \text{ in } W^{1,p}(R^{n})
\end{align*}

Therefore \( W_{m} \Rightarrow u \) in \( W^{1,p}(R^{n}) \).

Notice that \( W_{m} = u(1-J_{m}) \) is compactly supported since \( u \) is and \( W_{m} = 0 \) \text{ if } 0 < x_{n} < \frac{1}{m}.

We can then modify \( W_{m} \) (see Evans 630) to produce \( u_{m} \in C_{c}^{\infty}(R^{n}) \) such that \( u_{m} \Rightarrow u \text{ in } W^{1,p}(R^{n}) \) if \( u \in W^{1,p}(R^{n}) \).