AN INTRODUCTION TO DISPERSIVE PARTIAL DIFFERENTIAL EQUATIONS.

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Abstract. Lecture notes concerning basic properties of the solutions to the semi-linear Schrödinger and to the KdV equation. Based on these notes a series of lectures were given at the summer school in University of Texas, Austin, July 18-22, 2011.

1. Introduction.

A partial differential equation (PDE) is called dispersive if, when no boundary conditions are imposed, its wave solutions spread out in space as they evolve in time. As an example consider $iu_t + u_{xx} = 0$. If we try a simple wave of the form $u(x,t) = Ae^{i(kx-\omega t)}$, we see that it satisfies the equation if and only if $\omega = k^2$. This is called the dispersive relation and shows that the frequency is a real valued function of the wave number. If we denote the phase velocity by $v = \frac{\omega}{k}$ we can write the solution as $u(x,t) = Ae^{ik(x-v(k)t)}$ and notice that the wave travels with velocity k. Thus the wave propagates in such a way that large wave numbers travel faster than smaller ones. (Trying a wave solution of the same form to the heat equation $u_t - u_{xx} = 0$, we obtain that the ω is complexd valued and the wave solution decays exponential in time. On the other hand the transport equation $u_t - u_x = 0$ and the one dimensional wave equation $u_{tt} = u_{xx}$ are traveling waves with constant velocity.) If we add nonlinear effects and study $iu_t + u_{xx} = f(u)$, we will see that even the existence of solutions over small times requires delicate techniques.

Going back to the linear equation, consider $u_0(x) = \int_{\mathbb{R}} \hat{u}_0(k)e^{ikx}dk$. For each fixed k the wave solution becomes $u(x,t) = \hat{u}_0(k)e^{ik(x-kt)} = \hat{u}_0(k)e^{ikx}e^{-ik^2t}$. Summing over k (integrating) we obtain the solution to our problem

$$u(x,t) = \int_{\mathbb{R}} \hat{u}_0(k)e^{ikx-ik^2t}dk.$$

Since $|\hat{u}(k,t)| = |\hat{u}_0(k)|$ we have that $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Thus the conservation of the L^2 norm (mass conservation or total probability) and the fact that high frequencies travel faster, leads to the conclusion that not only the solution will disperse into separate waves but that its amplitude will decay over time. This is not anymore the case for solutions over compact domains. The dispersion is limited and for the nonlinear dispersive problems we notice a migration from low to high frequencies. This fact is captured by zooming more closely in the Sobolev norm

$$\|u\|_{H^s} = \sqrt{\int |\hat{u}(k)|^2(1+|k|)^{2s}dk}$$

and observing that it actually grows over time. To analyze further the properties of dispersive PDEs and outline some recent developments we start with a concrete example.

2. The semi-linear Schrödinger equation.

Consider the semi-linear Schrödinger equation (NLS) in arbitrary dimensions

$$\begin{cases} iu_t + \Delta u + \lambda |u|^{p-1}u = 0, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}^n). \end{cases} \tag{1}$$

for any $1 < p < \infty$. $H^s(\mathbb{R}^n)$ (the s Sobolev space) is a Banach space that contains all functions that along with their distributional $s-$derivatives belong to $L^2(\mathbb{R}^n)$. This norm is equivalent (through the basic properties of the Fourier transform) to

$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1+|\xi|)^{2s} d\xi \right)^{\frac{1}{2}} < \infty.$$

The restriction on the values of p for equation (1) will become apparent shortly. When $\lambda = -1$ the equation is called defocusing and when $\lambda = 1$ it is called focusing. NLS is a basic dispersive model that appears in nonlinear optics and water wave theory. We will study problems related to the NLS

i) of local-in-time nature (local existence of solutions, uniqueness, regularity),

ii) of global-in-time nature (existence for large times, finite time blow-up, scattering).

REMARKS

1. Consider X a Banach space. Starting with initial data $u_0 \in H^s(\mathbb{R}^n)$, we say that the solution exists locally-in-time, if there exists $T > 0$ and a subset X of $C^0_t H^s_x([0,T] \times \mathbb{R}^n)$ such that there exists a unique solution to (1). Note that if $u(x,t)$ is a solution to (1) then $-u(-x,t)$ is also a solution. Thus we can extend any solution in $C^0_t H^s_x([0,T] \times \mathbb{R}^n)$ to a solution in $C^0_t H^s_x([-T,T] \times \mathbb{R}^n)$. We also demand that there is continuity with respect to the initial data in the appropriate topology.

2. If T can be taken to be arbitrarily large then we say that we have a global solution. We will see that this is the case when $\lambda = -1$.

3. Assume $u_0 \in H^s(\mathbb{R}^n)$ and consider a local solution. If there is a $T^\star$ such that

$$\lim_{t \to T^\star} \|u(t)\|_{H^s} = \infty,$$

we say that the solution blows up in finite time. This property is usually proved along with the local theory and is referred to as the blow-up alternative. More precisely one often proves that if $(0, T^*)$ is the maximum interval of existence, then if $T^* < \infty$, we have $\lim_{t \to T^*} \|u(t)\|_{H^s} = \infty$. Analogous statements can be made for $(-T^*, 0)$. This is the case in the focusing problem when $\lambda = 1$.

4. As a Corollary to the blow-up alternative one obtains globally defined solutions if there is an a priori bound of the H^s norms for all times. This is of the form

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} < \infty,$$

and it usually comes from the conservation laws of the equation. For (1) this is usually the case for $s = 0,\ 1$. An *important* comment is in order. Our notion of global solutions in remark 2 does not require that $\|u(t)\|_{H^s}$ remains uniformly bounded in time. As we said unless $s = 0,\ 1$, it is not a triviality to obtain such a uniform bound. In case that we have quantum scattering, these uniform bounds are byproducts of the control we obtain on our solutions at infinity.

5. If $u_0 \in H^s(\mathbb{R}^n)$ and we have a well defined local solution, then for each $(0, T)$ we have that $u(t) \in H_x(\mathbb{R}^n)$. Regularity refers to the fact that if we consider $u_0 \in H^{s_1}(\mathbb{R}^n)$ with $s_1 > s$, then $u \in X \subset C_t^0 H_x^s([0, T_1] \times \mathbb{R}^n)$, with $T_1 = T$. Notice that any H^{s_1} solution is in particular an H^s solution and thus $(0, T_1) \subset (0, T)$. Regularity affirms that $T_1 = T$

and thus u cannot blow-up in H^{s_1} before it blows-up in H^s both backward and forward in time.

6. Scattering is the hardest problem of all. Assume that we have a globally defined solution (which is true for large data in the defocusing case). The problem then is divided into an easier (existence of the wave operator) and a harder (asymptotic completeness) problem. We will see shortly that the L^p norms of linear solutions decay in time. This time decay is suggestive that for large values of p the nonlinearity can become negligible as $t \to \pm\infty$. Thus we expect that u can be approximated by the solution of the linear equation. We have to add here that this theory is highly nontrivial for large data. For small data we can have global solutions and scattering even in the focusing problem.

7. A solution that will satisfy (at least locally) all these properties will be called a strong solution. We will give a more precise definition later in the notes. Let us only mention here that the equipment of our solutions will all these additional properties is of great importance. For example the fact that local H^1 solutions satisfy the energy conservation law is a byproduct not only of the local-in-time existence but also of the regularity and the continuity with respect to the initial data properties. The arguments are strong enough to prove that most of the times (local-in-time existence after all come from Banach's fixed point theorems) the data to solution map is uniformly continuous, even analytic in some cases.

The standard treatment of the subject is a the wonderful book of **Cazenave: Semilinear Schrödinger equations.** We will refer to this book throughout these notes. A more recent book by **Tao: Nonlinear dispersive equations** supplements the material that we are developing nicely.

We will mainly consider H^s solutions where s is an integer, but the question remains: For what values of $s \in \mathbb{R}$ one can expect reasonable solutions? The symmetries of the equation (1) can be very helpful.

First we have the **scaling symmetry**: Let $\lambda > 0$. If u is a solution to (1) then

$$u^\lambda(x,t) = \lambda^{-\frac{2}{p-1}} u(\frac{x}{\lambda},\ \frac{t}{\lambda^2}), \quad u_0^\lambda = \lambda^{-\frac{2}{p-1}} u_0(\frac{x}{\lambda}),$$

is a solution to the same equation. Then we have the **Galilean Invariance**: If u is a solution to (1) then

$$e^{ix\cdot v} e^{-it|v|^2} u(x - 2vt,\ t)$$

is a solution to the same equation with data $e^{ix\cdot v} u_0(x)$. There is also **time reversal symmetry** that we have talked about. **Spatial rotation symmetry** which leads to the property that if we start with radial initial data then we obtain a radially symmetric solution. **Time translation invariance** that leads for smooth solutions to the conservation of energy

$$E(u)(t) = \frac{1}{2} \int |\nabla u(t)|^2 dx - \frac{\lambda}{p+1} \int |u(t)|^{p+1} dx = E(u_0). \tag{2}$$

Phase rotation symmetry $e^{i\theta} u$ that leads to mass conservation

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \tag{3}$$

and **space translation invariance** that leads to the conservation of the momentum

$$\vec{p}(t) = \Im \int_{\mathbb{R}^n} \bar{u} \nabla u dx = \vec{p}(0). \tag{4}$$

In the case that $p = 1 + \frac{4}{n}$, we also have the **pseudo-conformal symmetry** where if u is a solution to (1) then for $t \neq 0$

$$\frac{1}{|t|^{\frac{n}{2}}} \overline{u(\frac{x}{t}, \frac{1}{t})} e^{\frac{i|x|^2}{4t}}$$

is also a solution. This leads to the pseudo-conformal conservation law

$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 - \frac{8t^2\lambda}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx = \|xu_0\|_{L^2}^2.$$

Let's go back to scaling for a moment. If we compute $\|u_0^\lambda\|_{\dot{H}^s}$ we see that

$$\|u_0^\lambda\|_{\dot{H}^s} = \lambda^{s_c - s} \|u_0\|_{\dot{H}^s}$$

where $s_c = \frac{n}{2} - \frac{2}{p-1}$. It is then clear that as $\lambda \to \infty$:

i) If $s > s_c$ (**sub-critical case**) the norm of the initial data can be made small while at the same time the time interval is made longer: this is the best possible scenario for local well-posedness. Notice that u^λ lives on $[0, \lambda^2 T]$.

ii) If $s = s_c$ (**critical case**) the norm of the initial data is invariant while the time interval gets longer: there is still hope but it turns out that to provide globally defined solutions one has to work very hard.

iii)If $s < s_c$ (**super-critical case**) the norms grow as the time interval is made longer: scaling is against us and indeed we cannot expect even locally defined solutions.

A lot of attention has recently been paid to concrete examples of ill-posed solutions when $s < s_c$. For the focusing problem one can use the specific solution (solitons etc) of the problem and show that the data to solution map is not uniformly continuous (see Kenig, Ponce, Vega). For defocusing equations the problem is a little harder. Christ, Colliander and Tao, among others, have recently demonstrated different modes of ill-posedness for defocusing equations.

3. Local Well-Posedness

The harder problem to resolve when looking for local solutions is to construct the aforementioned Banach space X. This process is delicate (the exception been of course the construction of smooth solutions that is done classically) and is build on certain estimates that the linear solution satisfies. Recall from our undergraduate (or graduate) PDE classes, we can obtain the solution to the linear problem by utilizing the Fourier transform. Then for smooth initial data (say in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$) the solution of the linear equation is given as the convolution of the data with the tempered distribution

$$K_t(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{i\frac{i|x|^2}{4t}}.$$

Thus we write for the solution

$$u(x,t) = U(t)u_0(x) = e^{it\Delta}u_0(x) = K_t \star u_0(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy. \tag{5}$$

Another fact from our undergraduate (or graduate) machinery is Duhamel's principle: Let I be any time interval and suppose that $u \in C_t^1\mathcal{S}(I \times \mathbb{R}^n)$ and that $F \in C_t^0\mathcal{S}(I \times \mathbb{R}^n)$. Then u solves

$$
\begin{cases} iu_t + \Delta u = F, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,t_0) = u(t_0) \in \mathcal{S}(\mathbb{R}^n) \end{cases} \tag{6}
$$

if and only if

$$
u(x,t) = e^{i(t-t_0)\Delta}u(t_0) - i\int_0^t e^{i(t-s)\Delta}F(s)ds. \tag{7}
$$

For less regular H^s data, a time interval I which contains zero, and

$$
F \in C(H^s(\mathbb{R}^n); H^{s-2}(\mathbb{R}^n)),
$$

we call $u \in C(I; H^s(\mathbb{R}^n)) \cap C^1(I; H^{s-2}(\mathbb{R}^n))$ a strong solution of (6) on I, if it satisfies the equation for all $t \in I$ in the sense of H^{s-2} (thus as a distribution for low values of s) and $u(0) = u_0$. By a little semigroup theory this definition of a strong solution is equivalent into say that for almost all $t \in I$, u satisfies (7).

We now state Strichartz's estimates. It is immediate from (5) that the linear solution satisfies

$$
\|u\|_{L^\infty_x} \leq \frac{1}{(4|t|\pi)^{\frac{n}{2}}}\|u_0\|_{L^1}.
$$

In addition the solution on the dual Fourier transform variables satisfies that $\hat{u}(\xi,t) = e^{-4\pi^2 it|\xi|^2}\hat{u}_0(\xi)$ and Plancherel's theorem implies that

$$
\|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x}.
$$

Riesz-Thorin interpolation Lemma then implies that for any $p \geq 2$ and $t \neq 0$ we have that

$$
\|u(t)\|_{L^p_x} \leq \frac{1}{(4|t|\pi)^{n(\frac{1}{2}-\frac{1}{p})}}\|u_0\|_{L^{p'}}
$$

where p' is the dual exponent of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

Fortunately we can extend these basic dispersive estimates by duality (using a $TT^\star$ argument) and obtain the famous Strichartz estimates. Strichartz, Ginibre and Velo, Keel and Tao are the relevant names here.

Theorem 1. *Fix $n \geq 1$. We call a pair (q,r) of exponents admissible if $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $(q,r,n) \neq (2,\infty,2)$. Then for any admissible exponents (q,r) and $(\tilde{q},\tilde{r})$ we have the following estimates: The linear estimate*

$$\|U(t)u_0\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L^2}, \tag{8}$$

and the nonlinear estimate

$$\|\int_0^t U(t-s)F(s)ds\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^n)} \tag{9}$$

where $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ and $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$.

Remark: Strichartz estimates actually give you more. In particular the operator $e^{it\Delta}u_0(x)$ belongs to $C(I, L_x^2)$ where I is any interval of $\mathbb{R}$ and $\int_0^t U(t-s)F(s)ds$ belongs to $C(\bar{I}, L_x^2)$.

We are now ready for a precise definition of what we mean by local well-posedness of the initial value problem (IVP) (1).

Definition 1. *We say that the IVP (1) is locally well-posed (lwp) and admits a strong solution in $H^s(\mathbb{R}^n)$ if for any ball B in the space $H^s(\mathbb{R}^n)$, there exists a finite time T and a Banach space $X \subset L_t^\infty H_x^s([0,T]\times\mathbb{R}^n)$ such that for any initial data $u_0 \in B$ there exists a unique solution $u \in X \subset C_t^0 H_x^s([0,T]\times\mathbb{R}^n)$ to the integral equation*

$$u(x,t) = U(t)u_0 - i\int_0^t U(t-s)|u|^{p-1}u(s)ds.$$

Furthermore the map $u_0 \to u(t)$ *is continuous as a map from* $H^s(\mathbb{R}^n)$ *into* $C^0_t H^s_x([0,T] \times \mathbb{R}^n)$. *If uniqueness holds in the whole space* $C^0_t H^s_x([0,T] \times \mathbb{R}^n)$ *then we say that the lwp is unconditional.*

In this case u satisfies the equation in $H^{s-2}(\mathbb{R}^n)$ for almost all $t \in [0,T]$. The details can be found in Cazenave's book. In what follows we assume that the nonlinearity is sufficiently smooth (eg p-1=2k).

• We start with the H^s well-posedness theory where $s > \frac{n}{2}$ is an integer.

Theorem 2. *Let* $s > \frac{n}{2}$ *an integer. For every* $u_0 \in H^s(\mathbb{R}^n)$ *there exists* $T^* > 0$ *and a unique maximal solution* $u \in C((0,T^*); H^s(\mathbb{R}^n))$ *that satisfies* (1) *and in addition satisfies the following properties:*

i) If $T^* < \infty$ *then* $\|u(t)\|_{H^s} \to \infty$ *as* $t \to \infty$. *Moreover* $\limsup \|u(t)\|_{L^\infty} = \infty$ *as* $t \to \infty$.

ii) u *depends continuously on the initial data in the following sense. If* $u_{n,0} \to u_0$ *in* H^s *and if* u_n *is the corresponding maximal solution with initial data* $u_{n,0}$, *then* $u_n \to u$ *in* $L^\infty((0,T); H^s(\mathbb{R}^n))$ *for every interval* $[0,T] \subset [0,T^*)$.

iii) In addition u *satisfies conservation of mass,* (3), *and conservation of energy,* (2).

Remark. A remark about uniqueness. Suppose that one proves existence and uniqueness in $C([-T,T]; X_M)$ where X_M, $M = M(\|u_0\|_X)$, $T = T(M)$, is a fixed ball in the space X. One can then easily extend the uniqueness to the whole space X by shrinking time by a fixed amount. Indeed, shrinking time to T' we get existence and uniqueness in a larger ball $X_{M'}$. Now assume that there are two different solutions one staying in the ball X_M and one separating after hitting the boundary at some time $|t| < T'$. This is already a contradiction by the uniqueness in $X_{M'}$.

To prove Theorem 2 we need the following two lemmas:

Lemma 1. *Gronwall's inequality: Let $T > 0$, $k \in L^1(0,T)$ with $k \geq 0$ a.e. and two constants $C_1, C_2 \geq 0$. If $\psi \geq 0$, a.e in $L^1(0,T)$, such that $k\psi \in L^1(0,T)$ satisfies*

$$\psi(t) \leq C_1 + C_2 \int_0^t k(s)\psi(s)ds$$

for a.e. $t \in (0,T)$ then,

$$\psi(t) \leq C_1 \exp\big(C_2 \int_0^t k(s)ds\big).$$

Lemma 2. *Let $g(u) = \pm|u|^{2k}u$ and consider and $s, l \geq 0$, integers with $l \leq s$ and $s > \frac{n}{2}$. Then*

$$\|g(u) - g(v)\|_{L^2} \lesssim \big(\|u\|_{H^s}^{2k} + \|v\|_{H^s}^{2k}\big)\|u - v\|_{L^2}. \tag{10}$$

$$\|g^{(l)}(u) - g^{(l)}(v)\|_{L^\infty} \lesssim \big(\|u\|_{H^s}^{2k-l} + \|v\|_{H^s}^{2k-l}\big)\|u - v\|_{H^s}. \tag{11}$$

$$\|g(u)\|_{H^s} \lesssim \|u\|_{H^s}^{2k+1}. \tag{12}$$

$$\|g(u) - g(v)\|_{H^s} \lesssim \big(\|u\|_{H^s}^{2k} + \|v\|_{H^s}^{2k}\big)\|u - v\|_{H^s}. \tag{13}$$

Proof. To prove (12) we use the algebra property and the fact that $\|u\|_{H^s} = \|\bar{u}\|_{H^s}$. To prove (10) and (11) note that since g is smooth we have that

$$|g(u) - g(v)| \lesssim \big(|u|^{2k} + |v|^{2k}\big)|u - v|,$$

$$|g^{(l)}(u) - g^{(l)}(v)| \lesssim \big(|u|^{2k-l} + |v|^{2k-l}\big)|u - v|.$$

Then

$$\|g(u) - g(v)\|_{L^2} \lesssim \big(\|u\|_{L^\infty}^{2k} + \|v\|_{L^\infty}^{2k}\big)\|u - v\|_{L^2} \lesssim \big(\|u\|_{H^s}^{2k} + \|v\|_{H^s}^{2k}\big)\|u - v\|_{L^2},$$

$$\|g^{(l)}(u) - g^{(l)}(v)\|_{L^\infty} \lesssim \big(\|u\|_{L^\infty}^{2k-l} + \|v\|_{L^\infty}^{2k-l}\big)\|u - v\|_{L^\infty} \lesssim \big(\|u\|_{H^s}^{2k-l} + \|v\|_{H^s}^{2k-l}\big)\|u - v\|_{L^2},$$

where we used the fact that H^s embeds in L^∞. To prove (13) notice that the L^2 part of the left hand side follows from (10). For the derivative part consider a multi-index α with $|\alpha| = s$. Then $D^\alpha u$ is the sum (over $k \in \{1, 2, ..., s\}$) of terms of the form $g^{(k)}(u) \prod_{j=1}^k D^{\beta_j} u$ where $|\beta_j| \geq 1$ and $|\alpha| = |\beta_1| + ... + |\beta_k|$. Now let $p_j = \frac{2s}{|\beta_j|}$ such that $\sum_{j=1}^k \frac{1}{p_j} = \frac{1}{2}$. We have by Hölder's inequality

$$\|g^{(k)}(u) \prod_{j=1}^k D^{\beta_j} u\|_{L^2} \lesssim \|g^{(k)}(u)\|_{L^\infty} \prod_{j=1}^k \|D^{\beta_j} u\|_{L^{p_j}}.$$

By complex interpolation (or Gagliardo-Nirenberg inequality) we obtain

$$\|D^{\beta_j} u\|_{L^{p_j}} \lesssim \|u\|_{H^s}^{\frac{|\beta_j|}{s}} \|u\|_{L^\infty}^{1-\frac{|\beta_j|}{s}}$$

and thus

$$\|g^{(k)}(u) \prod_{j=1}^k D^{\beta_j} u\|_{L^2} \lesssim \|g^{(k)}(u)\|_{L^\infty} \|u\|_{H^s} \|u\|_{L^\infty}^{k-1} \lesssim \|u\|_{H^s}^{2k+1}$$

where in the last inequality we used (11). Thus we obtain

$$\|D^\alpha u\|_{L^2} \lesssim \|u\|_{H^s}^{2k+1}. \tag{14}$$

Again notice that the term $D^\alpha(g(u) - g(v))$ is the sum of terms of the form

$$g^{(k)}(u) \prod_{j=1}^k D^{\beta_j} u - g^{(k)}(v) \prod_{j=1}^k D^{\beta_j} v = \left[g^{(k)}(u) - g^{(k)}(v)\right] \prod_{j=1}^k D^{\beta_j} u + g^{(k)}(v) \prod_{j=1}^k D^{\beta_j} w_j$$

where w_j's are equal to u or v except one that is equal to $u - v$. The second of the left hand side is estimated as in the proof of (14). For the first the same trick applies but now to estimate $\|g^{(k)}(u) - g^{(k)}(v)\|_{L^\infty}$ we use (13). □

It remains to prove Theorem 2. *Existence and Uniqueness.* We construct solutions by a fixed point argument. Given $M, T > 0$ to be chosen later, we set $I = (0, T)$ and consider

$$\mathcal{A} = \{u \in L^\infty(I; H^s(\mathbb{R}^n)) : \|u\|_{L^\infty(I;H^s)} \leq M\}.$$

(E, d) is a complete metric space where the distance is defined by $d(u, v) = \|u - v\|_{L^\infty(I;L^2)}$.

We now consider (equation (1) with $\lambda = -1$)

$$\Phi(u)(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}|u|^{2k}u(s)ds = e^{it\Delta}u_0 + H(u)(t).$$

By the Lemma 2, Minkowski's inequality and the fact that $e^{it\Delta}$ is an isometry in H^s we have that

$$\|\Phi(u)(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + T\|g(u)\|_{L^\infty(I;H^s)} \le TC(M)M.$$

Furthermore using the Lemma 2 again we have

$$\|\Phi(u)(t) - \Phi(v)(t)\|_{L^2} \lesssim TC(M)\|u - v\|_{L^\infty(I;L^2)}. \tag{15}$$

Therefore we see that if $M = 2\|u_0\|_{H^s}$ and $TC(M) < \frac{1}{2}$, then Φ is a contraction of (E, d) and thus has a unique fixed point. Uniqueness in the full space follows by the remark above.

Maximal solutions, blow-up alternative. Let $u_0 \in H^s$ and define

$$T^* = \sup(T > 0 : \textit{there exists a solution on } [0, T]). \tag{16}$$

Now let $T^* < \infty$ and assume that there exists a sequence $t_j \to T^*$ such that $\|u(t_j)\|_{H^s} \le M$. In particular for k such that t_k is close to T^* we have that $\|u(t_k)\|_{H^s} \le M$. Now we solve our problem with initial data $u(t_k)$ and we extend our solution to the interval $[t_k,\ t_k + T(M)]$. But if we pick k such that

$$t_k + T(M) > T^*$$

we then contradict the definition of T^*. Thus $\lim_{t\to T^*} \|u(t)\|_{H^s} = \infty$ if $T^* < \infty$. We now show that if $T^* < \infty$ then $\limsup \|u(t)\|_{L^\infty} = \infty$. Indeed suppose that $\limsup \|u(t)\|_{L^\infty} <$

∞. Since $u \in C([0,T^*);H^s)$ we have that

$$M = \sup_{0\leq t<T^*} \|u(t)\|_{L^\infty} < \infty$$

where we used the fact that H^s embeds in L^∞. By Duhamel's formula and Lemma 2 we have that

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + C(M)\int_0^t \|u(s)\|_{H^s} ds.$$

By Gronwall's lemma we have that $\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} e^{T^*C(M)}$ for all $0 \leq t < T^*$. But this contradicts the blow-up of $\|u(t)\|_{H^s}$ at T^*.

<u>*Continuous dependence:*</u> Let $u_0 \in H^s$ and consider $u_{0,n} \subset H^s$ such that $u_{n,0} \to u_0$ in H^s as $n \to \infty$. Since for n sufficiently large we have that $\|u_{0,n}\|_{H^s} \leq 2\|u_0\|_{H^s}$ by the local theory there exists $T = T(\|u_0\|_{H^s})$ such that u and u_n are defined on $[0,T]$ for $n \geq N$ and

$$\|u\|_{L^\infty((0,T);H^s)} + \sup_{n\geq N} \|u_n\|_{L^\infty((0,T);H^s)} \leq 4\|u_0\|_{H^s}.$$

Now note that $u_n(t) - u(t) = e^{it\Delta}(u_{n,0} - u_0) + H(u_n)(t) - H(u)(t)$. If we use Lemma 2 we see that for all $t \in (0,T)$ and n sufficiently large, there exists C such that

$$\|u_n(t) - u(t)\|_{H^s} \leq \|u_{n,0} - u_0\|_{H^s} + C\int_0^t \|u_n(s) - u(s)\|_{H^s} ds.$$

By Gronwall's lemma we see that $u_n \to u$ in H^s as $n \to \infty$. Iterating this property to cover any compact subset of $(0,T^*)$ we finish the proof.

As a final note we remark that if we solve the equation, starting from u_0 and $u(t_1)$ over the intervals $[0,t_1]$ and $[t_1,t_2]$ respectively, by continuous dependence, to prove that $C\big([0,T];H^s(\mathbb{R}^n)\big)$, it is enough to consider the difference $u(t_1) - u_0$ in the H^s norm. Since

$$u(t_1) - u_0 = (e^{it_1\Delta} - 1)u_0 - i\int_0^{t_1} e^{i(t_1-s)} g(u)(s) ds,$$

using again Lemma 2 and the fact that $e^{it\Delta}u_0(x) \in C(\mathbb{R}; H^s)$ we have

$$\|u(t_1) - u_0\|_{H^s} \lesssim \|(e^{it_1\Delta} - 1)u_0\|_{H^s} + |t_1|\|u\|^{2k+1}_{L^\infty((0,t_1);H^s)}$$

which finishes the proof.

Conservation laws: Since we develop the H^1 theory below we implicitly have $s \geq 2$. We have at hand a solution that satisfies the equation in the classical sense for high enough s (in general in the H^{s-2} sense with $s \geq 2$ and thus in particular u satisfies the equation at least in the L^2 sense. All integrations below then can be justified in the Hilbert space L^2). To obtain the conservation of mass we can multiply the equation by $\bar{u}$, integrate and then take the real part. To obtain the conservation of energy we multiply the equation by $\bar{u}_t$, take the real part and then integrate.

• We continue with the H^1 sub-critical theory (Kato, Ginibre and Velo).

Theorem 3. *For every $u_0 \in H^1(\mathbb{R}^n)$ there exists a unique strong solution of (1) defined on the maximal interval $(0, T_{\max})$. Moreover*

$$u \in L^\gamma_{loc}((0, T_{\max});\ W^{1,\rho}_x(\mathbb{R}^n))$$

for every admissible pair (γ, ρ). In addition

$$\lim_{t \to T_{\max}} \|u(t)\|_{H^1} = \infty$$

if $T_{\max} < \infty$, and u depends continuously on u_0 in the following sense: There exists $T > 0$ depending on $\|u_0\|_{H^1}$ such that if $(u_0)_n \to u_0$ in H^1 and $u_n(t)$ is the corresponding solution of (1), then $u_n(t)$ is defined on $[0, T]$ for n sufficiently large and

$$u_n(t) \to u(t) \quad in \quad C([0, T];\ H^1) \tag{17}$$

for every compact interval $[0,T]$ of $(0,T_{\max})$. Finally we have that

$$E(u)(t)=\frac{1}{2}\int|\nabla u(t)|^2dx-\frac{\lambda}{p+1}\int|u(t)|^{p+1}dx=E(u_0)$$

and

$$M(u)(t)=\|u(t)\|_{L^2}=\|u_0\|_{L^2}=M(u_0).$$

We note that $W^{1,\rho}$ is the Sobolev space with one weak derivative in L^ρ. We first write the solution map using Duhamel's formula as

$$\Phi(u)(t)=e^{it\Delta}u_0-i\int_0^t e^{i(t-s)\Delta}|u|^{p-1}u(s)ds$$

Now pick $r=p+1$. Fix $M,T>0$ to be chosen later and let q be such that the pair (q,r) is admissible (since the admissibility condition reads $\frac{2}{q}+\frac{n}{r}=\frac{n}{2}$ we have that $q=\frac{4(p+1)}{n(p-1)}$). We run a contraction argument on the set $\mathcal{A}=$

$$\Big(u\in L_t^\infty H_x^1([0,T]\times\mathbb{R}^n)\cap L^q((0,T);W^{1,r}(\mathbb{R}^n)):\|u\|_{L_t^\infty((0,T);H^1)}\leq M,\ \|u\|_{L_t^qW_x^{1,r}}\leq M.\Big)$$

Notice that for $r=p+1$ we have

$$\||u|^{p-1}u\|_{L_x^{r'}}\lesssim\|u\|_{L_x^r}^p$$

and thus

$$\||u|^{p-1}u\|_{L_t^qL_x^{r'}}\lesssim\|u\|_{L_t^\infty L_x^r}^{p-1}\|u\|_{L_t^qL_x^r}.$$

The last thing to notice is that since $p<1+\frac{4}{n-2}$ we have that $q>2$ and thus $q>q'$. By Sobolev embedding we also have that

$$\|u\|_{L_x^{p+1}}\lesssim\|u\|_{H^1}$$

and thus

$$\||u|^{p-1}u\|_{L_t^q L_x^{r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{p-1} \|u\|_{L_t^q L_x^r}.$$

Similarly since the nonlinearity is smooth

$$\|\nabla(|u|^{p-1}u)\|_{L_t^q L_x^{r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{p-1} \|\nabla u\|_{L_t^q L_x^r}$$

and all in all

$$\||u|^{p-1}u\|_{L_t^q W_x^{1,r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{p-1} \|u\|_{L_t^q W_x^{1,r}} \leq M^{p-1} \|u\|_{L_t^q W_x^{1,r}}.$$

Now using Hölder's inequality in time we have

$$\||u|^{p-1}u\|_{L_t^{q'} W_x^{1,r'}} \lesssim T^{\frac{q-q'}{q'q}} \||u|^{p-1}u\|_{L_t^q W_x^{1,r'}} \lesssim T^{\frac{q-q'}{q'q}} \|u\|_{L_t^\infty H_x^1}^{p-1} \|u\|_{L_t^q W_x^{1,r}}.$$

Now using Duhamel and Strichartz with two different pairs on the left hand side we have that

$$\|\Phi(u)(t)\|_{L_t^q W_x^{1,r}} \lesssim \|e^{it\Delta}u_0\|_{L_t^q W_x^{1,r}} + \||u|^{p-1}u\|_{L_t^{q'} W_x^{1,r'}} \lesssim$$

$$\|u_0\|_{H^1} + T^{\frac{q-q'}{q'q}} \|u\|_{L_t^\infty H_x^1}^{p-1} \|u\|_{L_t^q W_x^{1,r}}$$

and

$$\|\Phi(u)(t)\|_{L_t^\infty H_x^1} \lesssim \|u_0\|_{H^1} + T^{\frac{q-q'}{q'q}} \|u\|_{L_t^\infty H_x^1}^{p-1} \|u\|_{L_t^q W_x^{1,r}}.$$

Thus

$$\|\Phi(u)(t)\|_{L_t^q W_x^{1,r}} + \|\Phi(u)(t)\|_{L_t^\infty H_x^1} \leq C\|u_0\|_{H^1} + CT^{\frac{q-q'}{q'q}} M^{p-1} \|u\|_{L_t^q W_x^{1,r}}.$$

We can now pick T small such that given $\|u_0\|_{H^1}$, we set $M = 2C\|u_0\|_{H^1}$ and $CT^{\frac{q-q'}{q'q}} M^{p-1} \leq \frac{1}{2}$. For such $T \sim T(\|u_0\|_{H^1})$ we have that $\|\Phi(u)(t)\|_{\mathcal{A}} \leq M$ and thus $\Phi : \mathcal{A} \to \mathcal{A}$. The same estimate on the difference

$$\|\Phi(u)(t) - \Phi(v)(t)\|_{L_t^q W_x^{1,r}} + \|\Phi(u)(t) - \Phi(v)(t)\|_{L_t^\infty H_x^1}$$

provides a unique solution $u \in \mathcal{A}$. Notice that by the above estimates and the Strichartz estimates we have that $u \in C_t^0((0,T);H^1(\mathbb{R}^n))$. To extend uniqueness in the full space we assume that we have another solution v and consider an interval $[0,\delta]$ with $\delta < T$. Then as before

$$\|u(t)-v(t)\|_{L^q_\delta W^{1,r}_x} + \|u(t)-v(t)\|_{L^\infty_\delta H^1_x} \leq C\delta^\alpha(\|u\|^{p-1}_{L^\infty_T H^1_x} + \|v\|^{p-1}_{L^\infty_T H^1_x})\|u-v\|_{L^q_\delta W^{1,r}_x}$$

But if we set

$$K = \max(\|u\|_{L^\infty_T H^1_x} + \|v\|_{L^\infty_T H^1_x}) < \infty$$

then for δ small enough we obtain

$$\|u(t)-v(t)\|_{L^q_\delta W^{1,r}_x} + \|u(t)-v(t)\|_{L^\infty_\delta H^1_x} \leq \frac{1}{2}(\|u(t)-v(t)\|_{L^q_\delta W^{1,r}_x} + \|u(t)-v(t)\|_{L^\infty_\delta H^1_x})$$

which forces $u = v$ on $[0,\delta]$. To cover the whole $[0,T]$ we iterate the previous argument $\frac{T}{\delta}$ times.

• Continuous dependence is almost identical to the uniqueness argument we outlined and we will skip it.

• For the proofs of the conservation laws we refer to Cazenave's book.

• The fact that

$$u \in L^\gamma_{loc}((0,T^*);\; W^{1,\rho}_x(\mathbb{R}^n))$$

for every admissible pair (γ,ρ), follows from the Strichartz estimates on any compact interval inside $(0,T^*)$.

• What about T^* and maximality. The proof is the same as in the smooth case.

Remarks: 1. Notice that when $\lambda = -1$ (defocusing case), the mass and energy conservation provide a global a priori bound

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} \le C_{M(u_0),E(u_0)}.$$

By the blow-up alternative we then have that $T^* = \infty$ and the problem is globally well-posed (gwp).

2. Let $I = [0,T]$. An inspection of the proof reveals that we can run the lwp argument in the space $\mathcal{S}^1(I\times\mathbb{R}^n)$ with the norm

$$\|u\|_{\mathcal{S}^1(I\times\mathbb{R}^n)} = \|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} + \|\nabla u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)}$$

where

$$\|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} = \sup_{(q,r)-admissible} \|u\|_{L^q_{t\in I}L^r_x}.$$

- We now state the lwp theory which is due to Tsutsumi (1987) for the L^2 sub-critical problem.

Theorem 4. *Consider $1 < p < 1 + \frac{4}{n}$, $n \ge 1$ and an admissible pair (q,r) with $p+1 < q$. Then for every $u_0 \in L^2(\mathbb{R}^n)$ there exists a unique strong solution of*

$$\begin{cases} iu_t + \Delta u + \lambda |u|^{p-1}u = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{18}$$

defined on the maximal interval $(0, T_{\max})$ such that

$$u \in C^0_t((0,T_{\max}); L^2(\mathbb{R}^n)) \cap L^q_{loc}((0,T_{\max}); L^r(\mathbb{R}^n)).$$

Moreover

$$u \in L^{\gamma}_{loc}((0,T_{\max}); L^{\rho}(\mathbb{R}^n))$$

for every admissible pair (γ,ρ)*. In addition*

$$\lim_{t\to T_{\max}} \|u(t)\|_{L^2}=\infty$$

if $T_{\max}<\infty$ *and* u *depends continuously on* u_0 *in the following sense: There exists* $T>0$ *depending on* $\|u_0\|_{L^2}$ *such that if* $(u_0)_n\to u_0$ *in* L^2 *and* $u_n(t)$ *is the corresponding solution of* (18)*, then* $u_n(t)$ *is defined on* $[0,T]$ *for* n *sufficiently large and*

$$u_n(t)\to u(t) \ \ in \ \ L^\gamma_{loc}([0,T];\ L^\rho(\mathbb{R}^n)) \tag{19}$$

for every admissible pair (γ,ρ) *and every compact interval* $[0,T]$ *of* $(0,T_{\max})$*. Finally we have that*

$$M(u)(t)=\|u(t)\|_{L^2}=\|u_0\|_{L^2}=M(u_0) \ \ and\ thus\ T_{\max}=\infty. \tag{20}$$

Remarks: 1. Notice that global well-posedness follows immediately.

2. The equation makes sense in H^{-2} for almost all $t\in(0,T_{\max})$.

• Finally we state the L^2-critical lwp theory when $p=1+\frac{4}{n}$. We should mention that a similar theory holds for the H^1 critical problem ($p=1+\frac{4}{n-2}$). For dimensions $n=1,2$ the problem is always energy sub-critical.

Theorem 5. *Consider* $p=1+\frac{4}{n}$, $n\geq 1$*. Then for every* $u_0\in L^2(\mathbb{R}^n)$ *there exists a unique strong solution of*

$$\begin{cases} iu_t+\Delta u+\lambda|u|^{\frac{4}{n}}u=0,\\ u(x,0)=u_0(x)\end{cases} \tag{21}$$

defined on the maximal interval $(0,T_{\max})$ *such that*

$$u\in C^0_t((0,T_{\max});L^2(\mathbb{R}^n))\cap L^{p+1}_{loc}((0,T_{\max});L^{p+1}(\mathbb{R}^n)).$$

Moreover

$$u \in L^{\gamma}_{loc}((0,T_{\max}); L^{\rho}(\mathbb{R}^n))$$

for every admissible pair (γ,ρ). *In addition if* $T_{\max} < \infty$

$$\lim_{t\to T_{\max}} \|u(t)\|_{L^q_{loc}((0,T_{\max});L^r(\mathbb{R}^n))} = \infty$$

for every admissible pair (q,r) *with* $r \geq p+1$. u *also depends continuously on* u_0 *in the following sense: If* $(u_0)_n \to u_0$ *in* L^2 *and* $u_n(t)$ *is the corresponding solution of* (21), *then* $u_n(t)$ *is defined on* $[0,T]$ *for* n *sufficiently large and*

$$u_n(t) \to u(t) \quad in \quad L^q([0,T]);\ L^r(\mathbb{R}^n)) \tag{22}$$

for every admissible pair (q,r) *and every compact interval* $[0,T]$ *of* $(0,T_{\max})$. *Finally we have that*

$$M(u)(t) = \|u(t)\|_{L^2} = \|u_0\|_{L^2} = M(u_0) \quad for\ all\ t \in (0,T_{\max}). \tag{23}$$

Remarks: 1. Notice that the blow-up alternative is not in terms of the L^2 which is the conserved quantity of the problem. This is because the problem is critical and the time of local well-posedness depends not only on the norm but also on the profile of the initial data. On the other hand if we have a global Strichartz bound on the solution global well-posedness is guaranteed by the Theorem. We will see later that this global Strichartz bound is sufficient for proving scattering also.

2. It is easy to see that if $\|u_0\|_{L^2} < \mu$, for μ small enough, then by the Strichartz estimates

$$\|e^{it\Delta}u_0\|_{L^{p+1}_t L^{p+1}_x(\mathbb{R}\times\mathbb{R}^n)} < C\mu < \eta.$$

Thus for sufficiently small initial data $T_{\max} = \infty$ and after only one iteration we have global well-posedness for the focusing or defocusing problem. In addition we have that $u \in L^q_t(\mathbb{R}; L^r_x(\mathbb{R}^n))$ for every admissible pair (q, r) and thus we also have scattering for small data. But this is not true for large data as the following example shows.

Consider $\lambda > 0$. We know that there exists nontrivial solutions of the form

$$u(x,t) = e^{i\omega t}\phi(x)$$

where ϕ is a smooth nonzero solution of

$$-\Delta\phi + \omega\phi = |\phi|^{p-1}\phi$$

with $\omega > 0$. But

$$\|\phi\|_{L^r_x(\mathbb{R}^n)} \leq M$$

for every $r \geq 2$ and thus $u \notin L^q_t(\mathbb{R}; L^r_x(\mathbb{R}^n))$ for any $q < \infty$.

4. Decay estimates.

To study in more details the local or global solutions of the above problems we have to revisit the symmetries of the equation. We first write down the local conservation laws or the conservation laws in differentiable form. The differential form of the conservation law is more flexible and powerful as it can be localized to any given region of space-time by integrating against a suitable cut-off function or contracting against a suitable vector fields. One then does not obtain a conserved quantity but rather an almost conserved or a monotone quantity. Thus from a single conservation law one can generate a variety of useful estimates, which can constrain the direction of propagation of a solution or provide a decay estimate for the solution or for the different components of the solution (eg the low or high frequency part of the solution). We can also use these formulas to study the blow-up and concentration problems for the focusing NLS and the scattering problem for the defocusing NLS.

The question of scattering or in general the question of dispersion of the nonlinear solution is tied to weather there is some sort of decay in a certain norm, such as the L^p norm for $p > 2$. In particular knowing the exact rate of decay of various L^p norms for the linear solutions, it would be ideal to obtain estimates that establish similar rates of decay for the nonlinear problem. The decay of the linear solutions can immediately establish weak quantum scattering in the energy space but to estimate the linear and the nonlinear dynamics in the energy norm we usually looking for the L^p norm of the nonlinear solution to go to zero as $t \to \infty$.

Strichartz type estimates assure us that certain L^p norms going to zero but only for the linear part of the solution. For the nonlinear part we need to obtain general decay estimates on solutions of defocusing equations. The mass and energy conservation laws establish the boundedness of the L^2 and the H^1 norms but are insufficient to provide a decay for higher powers of Lebesgue norms. In these notes we provide a summary of recent results that demonstrate a straightforward method to obtain such estimates by taking advantage of the momentum conservation law (4).

We start with some notation. Define the mass density

$$\rho = \frac{1}{2}|u|^2$$

and the momentum vector

$$p_k = \Im(\bar{u}\nabla_k u).$$

By straightforward calculations it is not hard to see that the mass density and the momentum satisfy the following local conservation laws. More precisely we have

Local conservation of mass:

$$\partial_t \rho + div \vec{p} = 0, \tag{24}$$

and local momentum conservation

$$\partial_t p_k + \nabla_j \left(\delta_k^j \left(-\frac{\lambda(p-1)}{p+1} |u|^{p+1} - \Delta \rho \right) + \sigma_k^j \right) = 0 \tag{25}$$

where the tensor σ_{jk} is given by

$$\sigma_{jk} = 2\Re(\nabla_j u \nabla_k \bar{u}) = \frac{1}{\rho}(p_j p_k + \nabla_j \rho \nabla_k \rho).$$

We also record the so called Morawetz action

$$M_a(t) = \int_{\mathbb{R}^n} \nabla a \cdot \vec{p}\, dx = \int_{\mathbb{R}^n} \nabla_j a \ p^j \ dx.$$

This is not a conserved quantity but it is very helpful in obtaining decay estimates. The weight function $a(x)$ is usually given by $a(x) = |x|$ and thus $\nabla_j a = \frac{x_j}{|x|}$.

We contract the momentum conservation equation with the vector field $\vec{X} = \nabla a$.

$$X^k \ \partial_t p_k = -X^k \ \nabla_j \left(\delta_k^j \left(-\frac{\lambda(p-1)}{p+1} |u|^{p+1} - \Delta \rho \right) + \sigma_k^j \right)$$

$$\partial_t (X^k \ p_k) = -\nabla_j \left(X^k \ \delta_k^j (-\frac{\lambda(p-1)}{p+1} |u|^{p+1} - \Delta \rho) + X^k \ \sigma_k^j \right)$$

$$+(\nabla_j X^k) \delta_k^j \left(-\frac{\lambda(p-1)}{p+1} |u|^{p+1} - \Delta \rho \right) + (\nabla_j X^k) \sigma_k^j.$$

$$\partial_t M = \partial_t \int_{\mathbb{R}^n} p_k X^k dx =$$

$$\int_{\mathbb{R}^n} (\nabla_j X^j)(-\frac{\lambda(p-1)}{p+1}|u|^{p+1} - \Delta\rho)dx + \int_{\mathbb{R}^n} (\nabla_j X^k)\sigma_k^j dx =$$

$$\int_{\mathbb{R}^n} (divX)(-\frac{\lambda(p-1)}{p+1}|u|^{p+1} - \Delta\rho)dx + \int_{\mathbb{R}^n} (\nabla_j X^k)\sigma_k^j dx.$$

$$\nabla_j X^k = \frac{\delta_j^k |x|^2 - x_j x^k}{|x|^3}, \qquad divX = \frac{n-1}{|x|}.$$

$$\partial_t M = -\lambda(n-1)\frac{p-1}{p+1}\int_{\mathbb{R}^n} \frac{|u|^{p+1}}{|x|}\, dx +$$

$$(n-1)\int_{\mathbb{R}^n} \left(-\Delta(\frac{1}{|x|})\right) \rho\, dx + \int_{\mathbb{R}^n} (\nabla_j X^k)\sigma_k^j\, dx.$$

Recall

$$\sigma_{jk} = 2\Re(\nabla_j u \nabla_k \bar{u})$$

and thus

$$(\nabla_j X^k)\sigma_k^j = \frac{2}{|x|}\left(|\nabla u|^2 - \frac{1}{|x|^2}|(x\cdot\nabla)u|^2\right) \geq 0.$$

In particular for $n = 3$

$$-\Delta(\frac{1}{|x|}) = 4\pi\delta(x).$$

For the defocusing case and for all $n \geq 3$ (the distribution $-\Delta(\frac{1}{|x|})$ is positive then) one can obtain

$$\int_{\mathbb{R}_t}\int_{\mathbb{R}^n} \frac{|u(x,t)|^{p+1}}{|x|}dx \lesssim \sup_t \|u(t)\|^2_{\dot{H}^{\frac{1}{2}}}.$$

The Morawetz action is bounded using Hardy's inequality but we will come back to the right hand side of the last inequality later.

Rewrite this identity again:

$$\partial_t M = \int_{\mathbb{R}^n} \Delta a(x)\left(-\frac{\lambda(p-1)}{p+1}|u|^{p+1} - \Delta\rho\right)dx + \int_{\mathbb{R}^n} (\partial_j\partial^k a(x))\sigma_k^j dx$$

for a general weight function $a(x)$. The first to consider these identities was Morawetz for the Klein-Gordon equation but the exposition here is close in spirit to Lin and Strauss that derived these estimates for the NLS.

If we pick $a(x) = |x|^2$, then $\Delta a(x) = 2n$ and $\partial_j \partial_k a(x) = 2\delta_{kj}$. Then

$$\partial_t M = -\frac{2n\lambda(p-1)}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx + 2 \int_{\mathbb{R}^n} |\nabla u|^2 dx =$$

$$8\left(\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{\lambda}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx\right) + \frac{2\lambda}{p+1}\left(4 - n(p-1)\right)\int_{\mathbb{R}^n} |u|^{p+1} dx =$$

$$8E(u(t)) + \frac{2\lambda}{p+1}\left(4 - n(p-1)\right)\int_{\mathbb{R}^n} |u|^{p+1} dx. \tag{26}$$

Thus if we define the quantity

$$V(t) = \int_{\mathbb{R}^n} a(x)\rho(x) dx,$$

with $a(x) = |x|^2$, we have that

$$\partial_t V(t) = \int_{\mathbb{R}^n} a(x)\partial_t \rho(x) dx = -\int_{\mathbb{R}^n} a(x)\ \nabla \cdot \vec{p}\ dx = M(t)$$

using integration by parts. Thus

$$\partial_t^2 V(t) = 8E(u(t)) + \frac{2\lambda}{p+1}\left(4 - n(p-1)\right)\int_{\mathbb{R}^n} |u|^{p+1} dx.$$

Another useful calculation is the following. Set

$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 - \frac{8t^2\lambda}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx =$$

$$\|xu\|_{L^2}^2 + 4t^2\|\nabla u\|_{L^2}^2 - 4t\int_{\mathbb{R}^n} x \cdot p\ dx - \frac{8t^2\lambda}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx =$$

$$2\int_{\mathbb{R}^n} a(x)\rho(x) dx + 8t^2 E(u(t)) - 2t\int_{\mathbb{R}^n} \nabla a \cdot p\ dx =$$

$$2\int_{\mathbb{R}^n} a(x)\rho(x) dx + 8t^2 E(u_0) - 2t\int_{\mathbb{R}^n} \nabla a \cdot p\ dx$$

when $a(x) = |x|^2$. But

$$2\partial_t \int_{\mathbb{R}^n} a(x)\rho(x)dx = 2\int_{\mathbb{R}^n} \nabla a \cdot p \; dx$$

and thus

$$\partial_t K(t) = -2t \int_{\mathbb{R}^n} \partial_j a(x) \partial_t p^j dx + 16tE(u_0) = -2t\partial_t M(t) + 16tE(u_0).$$

If we use (26) we have that

$$\partial_t K(t) = -\frac{4\lambda t}{p+1}\left(4 - n(p-1)\right) \int_{\mathbb{R}^n} |u|^{p+1} dx.$$

Notice that for $p = 1 + \frac{4}{n}$, the quantity $K(t)$ is conserved.

So far we have obtained the so called one particle inequalities. To obtain two particle inequalities we form the tensor product

$$u(x_1, x_2, t) = u_1(x_1, t)u_2(x_2, t) := u_1 u_2$$

with $(x_1, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$.

Tensor product satisfies

$$iu_t + \Delta_{2n} u = f(u),$$

$$f(u) = |u_1|^2 u + |u_2|^2 u.$$

Δ_{2n} the Laplacian in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n, \quad \nabla = (\nabla_{x_1}, \nabla_{x_2})$.

$$\Delta_{2n} = \nabla \cdot \nabla = (\nabla_{x_1}, \nabla_{x_2}) \cdot (\nabla_{x_1}, \nabla_{x_2}) = \Delta_{x_1} + \Delta_{x_2}.$$

$$\rho = \frac{1}{2}|u|^2 = 2\rho_1 \rho_2.$$

Local conservation of mass

$$\partial_t \rho + \nabla \cdot \Im(\bar{u}\nabla u) = 0$$

where

$$\nabla \cdot \Im(\bar{u}\nabla u) = (\nabla_x,\ \nabla_y) \cdot (2\rho(y)\vec{p}(x),\ 2\rho(x)\vec{p}(y)) = 2\rho(y)\nabla_x\vec{p}(x) + 2\rho(x)\nabla_y\vec{p}(y),$$

with $p(x) = \Im(\bar{u}(x)\nabla_x u)$.

Local conservation of momentum

$$\partial_t p_k + \nabla_j \left(\delta_j^k(\Phi(\rho) - \Delta\rho) + [\sigma_k^j]\right) = 0$$

with

$$\Phi(\rho) = 4\rho_1\rho_2(\rho_1 + \rho_2), \qquad p_k = \Im(\bar{u}\nabla_k u),$$

$$[\sigma_{kj}] = \begin{bmatrix} \sigma_{kj} & \sigma_{k'j} \\ \sigma_{kj'} & \sigma_{k'j'} \end{bmatrix}$$

and

$$\sigma_{kj} = 2\Re(\nabla_j^x u\overline{\nabla_k^x u}), \qquad \sigma_{k'j} = 2\Re(\nabla_k^y u\overline{\nabla_j^x u}),$$

$$\sigma_{kj'} = 2\Re(\nabla_k^x u\overline{\nabla_j^y u}), \qquad \sigma_{k'j'} = 2\Re(\nabla_k^y u\overline{\nabla_j^y u}).$$

Notice now that

$$\nabla_j[\sigma_k^j] = (\nabla_j^x,\ \nabla_j^y) \begin{bmatrix} \sigma_{kj} & \sigma_{k'j} \\ \sigma_{kj'} & \sigma_{k'j'} \end{bmatrix} = (\nabla_j^x\sigma_k^j + \nabla_j^y\sigma_k^{j'},\ \nabla_j^x\sigma_{k'}^j + \nabla_j^y\sigma_{k'}^{j'}).$$

Now let's consider an example that is inspired by A. Hassel and T. Tao. Consider $\vec{x} = (\vec{x}_1, \vec{x}_2) \in \mathbb{R}^6$.

Define the distance of $\vec{x}$ from the average of its components $(\frac{\vec{x}_1+\vec{x}_2}{2}, \frac{\vec{x}_1+\vec{x}_2}{2})$. Thus

$$d(\vec{x}) = |\vec{x}_1 - \vec{x}_2|.$$

The new vector field satisfies,

$$X^j = \nabla^j d$$

$$div\vec{X} = \frac{4}{d} = \frac{4}{|x_1 - x_2|}$$

$$-\Delta(div\vec{X}) = -\Delta_{x_1}(\frac{4}{d}) - \Delta_{x_2}(\frac{4}{d}) = 32\pi\delta(\vec{x_1} - \vec{x_2}).$$

The new interaction Morawetz for the tensor product now reads

$$M_a(t) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \vec{X} \cdot \Im(\overline{u}\nabla u)\ dx.$$

Thus differentiation in time yields

$$\partial_t \int_{\mathbb{R}^6} p_k X^k dx = \int_{\mathbb{R}^6} (div\vec{X})(\Phi(\rho) - \Delta\rho)dx + \int_{\mathbb{R}^6} (\nabla_j X^k)[\sigma_k^j]dx.$$

The term

$$\int_{\mathbb{R}^6} (div\vec{X})\Phi(\rho)dx$$

is easily seen to be positive since the nonlinearity is defocusing. On the other hand a "hard" calculation can show that

$$\int_{\mathbb{R}^6} (\nabla_j X^k)[\sigma_k^j]dx \geq 0.$$

As for the last term in case that $u_1 = u_2 = u$

$$\int_{\mathbb{R}^6} \delta(x_1 - x_2)|u(x_1,t)|^2|u(x_2,t)|^2\ dx = \int_{\mathbb{R}^3} |u(x,t)|^4\ dx.$$

Integrate in time

$$\int_{\mathbb{R}_t}\int_{\mathbb{R}^3} |u(x,t)|^4 dx \lesssim \sup_t |M(t)|.$$

$$|M(t)| = |\int_{\mathbb{R}^6} \vec{p} \cdot \vec{X}\ dx| = \lesssim \int_{\mathbb{R}^6} |\vec{p}|\ dx \lesssim$$

$$\left(\int_{\mathbb{R}^3} \rho(x_1)dx_1\right)\int_{\mathbb{R}^3} |u(x_2)||\nabla u(x_2)|dx_2.$$

Thus

$$|M(t)| \lesssim \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2$$

and

$$\int_{\mathbb{R}_t}\int_{\mathbb{R}^3}|u(x,t)|^4dx \lesssim \sup_t\left(\|u(t)\|_{L^2}^2\|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2\right).$$

IDEA: Tensor product in $3+3$ dimensions. Through the estimate $-\Delta(\frac{1}{|x|}) = 4\pi\delta(x)$ we obtain an estimate in $(3+3)-3=3$ dimensions.

Colliander, Holmer, Visan, Zhang applied this idea in dimension 1.

1d estimate: Tensor product of 4 solutions. Obtain a solution in $1+1+1+1$ dimensions.

"Delta reduction" will give an estimate in $(1+1+1+1)-3$ dimension.

The new vector field is the gradient of the distance function to the diagonal $(x,x,x,x)\in\mathbb{R}^4$.

One obtains

$$\int_{\mathbb{R}_t}\int_{\mathbb{R}}|u(x,t)|^8dx \lesssim \sup_t\left(\|u(t)\|_{L^2}^7\|u(t)\|_{\dot{H}^1}\right).$$

Colliander, Grillakis, Tz. applied the idea to a tensor product of two solutions in 2d. We obtained a solution in $2+2$ dimensions. "Delta reduction" will give an estimate in $(2+2)-3=1$ dimension. We then upgraded this estimate in 2d.

Consider in $\mathbb{R}^2$ the line that passes through the origin and has direction given by the unit vector,

$$L(\vec{0},\vec{\omega}) = \{\vec{x}(l) = l\vec{\omega}\}.$$

We can lift this line onto a diagonal of $\mathbb{R}^4$ as follows

$$\tilde{L}(\omega) = \{\vec{x}_1 = \vec{x}_2 = \vec{x}(l)\}.$$

Consider now the distance of a point

$$\vec{y} = (\vec{x}_1,\vec{x}_2)\in\mathbb{R}^2\times\mathbb{R}^2 : \vec{y} = (y_1,y_2,y_3,y_4)\in\mathbb{R}^4$$

from the diagonal of $\mathbb{R}^4$

$$d = \sqrt{\frac{(y_1 - y_3)^2}{2} + y_2^2 + y_4^2}.$$

Set $X^j = \nabla^j d, \quad s = \frac{y_1 - y_3}{\sqrt{2}}$, and $\nabla_s = \frac{\nabla_{y_1} - \nabla_{y_3}}{\sqrt{2}}$.

In the 2d defocusing case ignoring the nonlinearity (notice that our calculations are true for any $p > 1$) we have

$$\partial_t M \geq \int_{\mathbb{R}^4} -\Delta(div\vec{X})\rho \, dx + \int_{\mathbb{R}^4} (\nabla_j X^k)\sigma_k^j dx.$$

$$div\vec{X} = \frac{2}{d} \geq 0.$$

If $z = (s, y_2, y_4)$

$$(\nabla_k X^j)\sigma_j^k = \frac{2}{d}\left(|\nabla_z u|^2 - \frac{1}{|z|^2}|(z \cdot \nabla_z)u|^2\right) \geq 0.$$

$$-\Delta(div\vec{X}) = -\Delta_z(\frac{2}{|z|}) = 8\pi\delta(\vec{z}) = 8\pi\delta(\vec{x_1} = \vec{x_2} = \vec{x}(l)).$$

Finally

$$\int_{\mathbb{R}}\int_L |u(\vec{x}(l), t)|^4 dldt \lesssim \sup_t \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2.$$

Averaging

$$\sup_{x_0} \int_{\mathbb{R}_t \times \mathbb{R}^2} \frac{|u(x,t)|^4}{|x - x_0|} dxdt \lesssim \sup_t \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2$$

The estimate is useful for radial solutions. In particular we can obtain a global $L_t^6 L_x^6$ estimate by radial Sobolev.

It is time to show the complete calculations of the interaction Morawetz estimates in all dimensions. Consider the Morawetz action for the tensor product of solutions.

$$M(t) = \int_{\mathbb{R}^n \otimes \mathbb{R}^n} \nabla a(x) \cdot \Im\left(\overline{u_1 \otimes u_2}(x) \nabla(u_1 \otimes u_2(x))\right) dx$$

Here $u := (u_1 \otimes u_2)(t, x)$ where $x = (x_1, x_2) \in \mathbb{R}^n \otimes \mathbb{R}^n$. Take $u_1 = u_2$, and $a(x) = |x_1 - x_2|$ for $n \geq 2$ and observe that

$$\partial_{x_1} a(x_1, x_2) = \frac{x_1 - x_2}{|x_1 - x_2|} = -\frac{x_2 - x_1}{|x_1 - x_2|} = -\partial_{x_2} a(x_1, x_2)$$

Thus the action becomes

$$\int_{\mathbb{R}^n \otimes \mathbb{R}^n} \frac{x_1 - x_2}{|x_1 - x_2|} \cdot \{\vec{p}(x_1, t)\rho(x_2, t) - \vec{p}(x_2, t)\rho(x_1, t)\} dx_1 dx_2$$

and renaming $x_1 = x$ and $x_2 = y$ we define

$$D^{-(n-1)} f(x) = \int_{\mathbb{R}^n} \frac{1}{|x - y|} f(y) dy$$

Then

$$M(t) = \langle [x; D^{-(n-1)}]\rho(t) \mid \vec{p}(t)\rangle.$$

The new vector field is given by $\vec{X} = [x; D^{-(n-1)}]$ or

$$\vec{X} f(x) = \int_{\mathbb{R}^n} \frac{x - y}{|x - y|} f(y) dy.$$

The crucial property is that the derivatives of this operator $\partial_j X^k$ form a positive definite operator

$$\partial_j X^k = D^{-(n-1)} \delta_j^k + [x^k; R_j]$$

where R_j is the singular integral operator corresponding to the symbol $\frac{\xi_j}{|\xi|^{n-1}}$.

$$R_j = \partial_j D^{-(n-1)}$$

or

$$R_j f(x) = -\int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^3} f(y) dy.$$

The commutator $[x_k; R_j]$ acts on functions

$$[x_k; R_j]f(x) = \int_{\mathbb{R}^n} r_{jk}(x,y)f(y)dy$$

where

$$r_{kj}(x,y) = -\frac{(x_k - y_k)(x_j - y_j)}{|x-y|^3}.$$

Thus

$$(\partial_j X^k)f(x) = \int_{\mathbb{R}^n} \eta_j^k(x,y)f(y)dy$$

where

$$\eta_{kj}(x,y) = \frac{\delta_{kj}|x-y|^2 - (x_j - y_j)(x_k - y_k)}{|x-y|^3}.$$

Note that

$$\nabla \cdot \vec{X} = \partial_j X^j = nD^{-(n-1)} + [x^j; R_j] = (n-1)D^{-(n-1)}$$

Differentiate

$$M(t) = \langle [x; D^{-(n-1)}]\rho(t) \mid \vec{p}(t)\rangle = \langle \vec{X}\rho(t) \mid \vec{p}(t)\rangle.$$

$$\partial_t M(t) = \langle \vec{X}\partial_t\rho(t) \mid \vec{p}(t)\rangle - \langle \vec{X}\cdot\partial_t\vec{p}(t) \mid \rho(t)\rangle$$

where we use the fact that $\vec{X}$ is anti-symmetric. Recall

$$\partial_t\rho + \partial_j p^j = 0$$

and

$$\partial_t p_k + \nabla_j\left(\delta_k^j\left(-\frac{\lambda(p-1)}{p+1}|u|^{p+1} - \Delta\rho\right) + \sigma_k^j\right) = 0.$$

Since we do not want to carry over the coefficient $\frac{p-1}{p+1}$ we consider the $p=3$ cubic case.

$$\partial_t M(t) = \langle \sigma_k^j(t) \mid (\partial_j X^k)\rho(t)\rangle - \langle p^j(t) \mid (\partial_j X^k)p_k(t)\rangle +$$

$$\langle(-\Delta\rho(t) + 2\rho^2(t)) \mid (\partial_j X^j)\rho(t)\rangle$$

Now recalling that

$$\sigma_{jk} = \frac{1}{\rho}(p_j p_k + \partial_j \rho \partial_k \rho)$$

we have that

$$\partial_t M(t) = P_1 + P_2 + P_3 + P_4$$

where

$$P_1 := \langle \rho^{-1} \partial_k \rho \partial^j \rho \mid (\partial_j X^k)\rho(t)\rangle,$$

$$P_2 := \langle \rho^{-1} p_k p^j \mid (\partial_j X^k)\rho(t)\rangle - \langle p^j \mid (\partial_j X^k) p_k\rangle,$$

$$P_3 := \langle (-\Delta\rho) \mid (\partial_j X^j)\rho(t)\rangle = \langle (-\Delta\rho) \mid (\nabla \cdot \vec{X})\rho(t)\rangle,$$

$$P_4 := 2\langle \rho^2 \mid (\partial_j X^j)\rho(t)\rangle = 2\langle \rho^2 \mid (\nabla \cdot \vec{X})\rho(t)\rangle.$$

But

$$P_3 = \langle (-\Delta\rho) \mid (\nabla \cdot \vec{X})\rho(t)\rangle = (n-1)\langle (D^2\rho) \mid D^{-(n-1)}\rho(t)\rangle.$$

Thus

$$P_3 = \frac{n-1}{4}\|D^{-\frac{n-3}{2}}(|u|^2)\|_{L^2}^2.$$

For P_4

$$P_4 = 2\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\rho^2(x)\rho(y)}{|x-y|} dxdy \geq 0$$

To prove positivity for P_2 requires more work. Recall

$$P_2 = \int_{\mathbb{R}^n \times \mathbb{R}^n} \left(\frac{\rho(y)}{\rho(x)} p_k(x) p_j(y) - p_k(y) p_j(x) \right) \eta_{kj}(x,y) dxdy.$$

Thus if we define the two point momentum vector

$$\vec{J}(x,y) = \sqrt{\frac{\rho(y)}{\rho(x)}}\vec{p}(x) - \sqrt{\frac{\rho(x)}{\rho(y)}}\vec{p}(y)$$

we can write

$$P_2 = \frac{1}{2}\langle J^j J_k \mid (\partial_j X^k)\rangle \geq 0$$

since $\partial_j X^k$ is positive definite. We keep only P_3

$$\|D^{-\frac{n-3}{2}}(|u|^2)\|^2_{L^2_t L^2_x} \lesssim \|u\|^2_{L^\infty_t \dot{H}^{\frac{1}{2}}} \|u\|^2_{L^\infty_t L^2_x}.$$

For $n = 2$ the estimate reads

$$\|D^{\frac{1}{2}}(|u|^2)\|_{L^2_t L^2_x} \lesssim \|u\|_{L^\infty_t \dot{H}^{\frac{1}{2}}} \|u\|_{L^\infty_t L^2_x},$$

and by Sobolev embedding we have that

$$\|u\|^2_{L^4_t L^8_x} = \||u|^2\|_{L^2_t L^4_x} \lesssim \|D^{\frac{1}{2}}(|u|^2)\|_{L^2_t L^2_x}$$

$$\lesssim \|u\|_{L^\infty_t \dot{H}^{\frac{1}{2}}} \|u\|_{L^\infty_t L^2_x} \lesssim C_{E(u_0),M(u_0)}.$$

A simple variant in 1d gives

$$\|\partial_x(|u|^2)\|_{L^2_t L^2_x} \lesssim \|u\|^{\frac{1}{2}}_{L^\infty_t \dot{H}^1_x} \|u\|^{\frac{3}{2}}_{L^\infty_t L^2_x}$$

and

$$\|u\|^{p+3}_{L^{p+3}_t L^{p+3}_x} \lesssim \|u\|^3_{L^\infty_t L^2_x} \|u\|_{L^\infty_t \dot{H}^1_x}.$$

Notice the two terms that are obviously positive are the terms P_3 and P_4. We write them again and keep in mind that $n \geq 2$.

$$P_4 \sim \int\int |\rho(x)|^{\frac{p+1}{2}} \Delta a(x-y)\rho(y)dxdy$$

and

$$P_3 \sim \int\int -\Delta\rho(x)\Delta a(x-y)\rho(y)dxdy \sim \int\int -\Delta\rho(x)\frac{1}{|x-y|}\rho(y)dxdy =$$

$$\int -\Delta\rho(x)\left(\int \frac{1}{|x-y|}\rho(y)dy\right)dx = \int -\Delta\rho(x)D^{-(n-1)}\rho(x)dx =$$

$$\langle D^2\rho,\ D^{-(n-1)\rho}\rangle = \|D^{-\frac{n-3}{2}}\rho\|_{L^2}^2.$$

For $n = 1$ we have that $\Delta a(x,y) = \Delta|x-y| = 2\delta(x-y)$ and thus

$$P_3 =\sim \int\int -\Delta\rho(x)\delta(x-y)\rho(y)dxdy = \int \nabla\rho(x)\nabla\rho(x)dx = \|\nabla\rho\|_{L^2}^2$$

and

$$P_4 \sim\sim \int\int |\rho(x)|^{\frac{p+1}{2}}\delta(x-y)\rho(y)dxdy \sim \int |u(x,t)|^{p+3}dx.$$

5. Applications.

5.1. **Blow-up for the energy sub-critical problem.** We show a criterion for blow-up for the energy subcritical focusing problem ($\lambda = 1$) which is due to Zakharov and Glassey. We further restrict the power of the nonlinearity to $p \geq 1 + \frac{4}{n}$. Notice that in addition we assume that our data have some decay. From the lwp theory we have a well-defined solution in $(-T_{\min}, T_{\max})$.

$$\begin{cases} iu_t + \Delta u + |u|^{p-1}u = 0, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n,\ |x|^2dx), \end{cases} \tag{27}$$

for any $1 < p < 1 + \frac{4}{n-2}, \quad n \geq 3$ ($1 < p < \infty$ for $n = 1, 2$).

Recall that if

$$V(t) = \frac{1}{2}\int_{\mathbb{R}^n} |x|^2|u(x,t)|^2dx$$

then $\partial_t V(t) = M(t)$ where

$$M(t) = 2\int_{\mathbb{R}^n} \vec{x}\cdot\vec{p}\,dx = 2\int_{\mathbb{R}^n} \vec{x}\cdot\Im(\bar{u}\nabla u)\ dx$$

But from (26) we have that

$$\partial_t M(t) = 8E(u(t)) + \frac{2}{p+1}\left(4 - n(p-1)\right)\int_{\mathbb{R}^n} |u|^{p+1}dx$$

and thus

$$\partial_t^2 V(t) = 8E(u(t)) + \frac{2}{p+1}\left(4 - n(p-1)\right)\int_{\mathbb{R}^n} |u|^{p+1}dx \leq 8E(u_0)$$

where we used conservation of energy and the fact that $p \geq 1 + \frac{4}{n}$. Integrating twice we obtain that

$$V(t) \leq \theta(t) = 4t^2 E(u_0) + tM(0) + V(0) =$$

$$4t^2 E(u_0) + 2t\int_{\mathbb{R}^n} \vec{x} \cdot \Im(\bar{u_0}\nabla u_0)\ dx + \frac{1}{2}\|xu_0\|_{L^2}^2$$

Notice that the right hand side is a second degree polynomial and since

$$u_0 \in \Sigma = H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n,\ |x|^2 dx),$$

the coefficients of t are finite. Now if the initial data have negative energy, that is if

$$E(u_0) < 0,$$

the coefficient of t^2 is negative. On the other hand for all times

$$V(t) = \frac{1}{2}\int_{\mathbb{R}^n} |x|^2 |u(x,t)|^2 dx \geq 0.$$

Thus $V(t)$ starts at positive values $V(0)$ and at some finite time the second order polynomial $V(t)$ will cross the horizontal axis. Thus both $T_{\min}$ and $T_{\max}$ are finite. By the blow-up alternative of the lwp theory this gives that

$$\lim_{t\to T_{\max}} \|u(t)\|_{H^1} = \infty,$$

if in addition to $u_0 \in H^1$, we have that $\|xu_0\|_{L^2} < \infty$ and $E(u_0) < 0$. *Remarks.* 1. Note that the assumption $E(u_0) < 0$ is a sufficient condition for finite-time blow-up but it is not necessary. One can actually prove that for any $E_0 > 0$ there exists u_0 with $E(u_0) = E_0$ and $T_{\max} < \infty$. For details look at Cazenave's book.

2. If one assumes radial data and dimensions $n \geq 3$, then one can remove the restriction of finite variance on the initial data. This is due to Ogawa and Tsutsumi. A weaker result holds for $n = 2$. For further results one has to consult the papers of Martel, Merle, and Raphael.

3. Many results have been devoted to the rate of the blow-up for the focusing problem. A variant of the local well-posedness theory provides the following result: If $u_0 \in H^1$ and $T_{\max} < \infty$, then there exists a $\delta > 0$ such that for all $0 \leq t < T_{\max}$ we have that

$$\|\nabla u(t)\|_{L^2} \geq \frac{\delta}{(T_{\max} - t)^{\frac{1}{p-1} - \frac{n-2}{4}}}.$$

Note that the above gives a lower estimate but not an upper estimate. Merle and Raphael have provide an upper estimate for the L^2-critical case that is very close to the one above. We also have to mention here that in general the aforementioned blow-up rate is not optimal. Bourgain and Wang have constructed solutions that blow-up twice as fast.

5.2. **Global Well-Posedness and Blow-up for the L^2-critical problem.** We have seen that in the case that $p = 1 + \frac{4}{n}$ the local existence time depends not only on the norm of the initial data but also on the profile. This prevents the use of the conservation of mass law in order to extend the solutions globally, even in the defocusing case ($\lambda = -1$).

But for $\lambda < 0$ the conjecture is that $T_{\max} = \infty$.

In these notes we present a partial answer to the problem in the case that we start with large L^2 data but in addition we assume finite variance. Recall that

$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 + \frac{8t^2}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx$$

is a conserved quantity for $p = 1 + \frac{4}{n}$. Thus

$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 + \frac{8t^2}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx = \|xu_0\|_{L^2}^2.$$

It can be proved that if we start with initial data in the Σ class then the solution that is defined in $u \in C_t^0((0,T_{\max}); H_x^1)$ also satisfies that $u \in C_t^0((0,T_{\max}); \Sigma)$ and thus the first term of $K(t)$ is well-defined. The conservation law for $K(t)$ implies that

$$\frac{8t^2}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx \leq C$$

and thus

$$\int_{\mathbb{R}^n}|u|^{2+\frac{4}{n}}dx \leq \frac{1}{t^2}$$

for all $t \in (0, T_{\max})$. Thus if $T_{\max} < \infty$ one can integrate the above quantity from any $t < T_{\max}$ to $T_{\max}$ and obtain that

$$\int_t^{T_{\max}}\int_{\mathbb{R}^n}|u(x,t)|^{2+\frac{4}{n}}dxdt < C.$$

Since on the other hand we have that

$$u \in L_t^{2+\frac{4}{n}}((0,t); L_x^{2+\frac{4}{n}})$$

we conclude

$$L_t^{2+\frac{4}{n}}((0,T_{\max}); L_x^{2+\frac{4}{n}}) < \infty.$$

But this contradicts the blow-up alternative for this problem and thus $T_{\max} = \infty$. Actually since the L^2 Strichartz norm $L_t^{2+\frac{4}{n}} L_t^{2+\frac{4}{n}}$ is bounded we also have scattering.

Remark. Recently the conjecture was settled in dimensions $n \geq 2$ for radial data in a series of papers by Killip, Tao, Visan, and Zhang. The 1d case and the non-radial case it is still open even in H^1.

Let's derive now a **global well-posedness condition for the focusing equation**

$$iu_t + \Delta u + |u|^{\frac{4}{n}} u = 0, \tag{28}$$

that is due to Weinstein. We have already seen that for small enough L^2 data the problem, focusing or defocusing, has global solutions. Moreover if one assumes small L^2 (but not arbitrarily small) data which in addition are in H^1, global well-posedness follows by the Gagliardo-Nirenberg inequality. More precisely since

$$\|u(t)\|_{L^{2+\frac{4}{n}}}^{2+\frac{4}{n}} \leq C\|\nabla u(t)\|_{L^2}^2 \|u(t)\|_{L^2}^{\frac{4}{n}} = C\|\nabla u(t)\|_{L^2}^2 \|u_0\|_{L^2}^{\frac{4}{n}},$$

one can easily see that the energy functional

$$E(u)(t) = \frac{1}{2}\int |\nabla u(t)|^2 dx - \frac{1}{2+\frac{4}{n}} \int |u(t)|^{2+\frac{4}{n}} dx$$

is lowered bound by

$$E(u(t)) = E(u_0) \geq \|\nabla u(t)\|_{L^2}^2 \left(\frac{1}{2} - C\|u_0\|_{L^2}^{\frac{4}{n}}\right).$$

Thus for $\|u_0\|_{L^2} < \eta$, η a fixed number, we have that

$$\|\nabla u(t)\|_{L^2} + \|u(t)\|_{L^2} \leq C_{M(u_0),E(u_0)} < \infty.$$

By the blow-up alternative of the H^1 theory we see that $T_{\max} = \infty$.

The question remains what is the optimum η. It was conjectured that even with pure L^2 data the optimum η will be the mass of the ground state Q. Q is the solution to the elliptic equation

$$-Q + \Delta Q = |Q|^{\frac{4}{n}} Q = 0,$$

which follows from (28) using the ansatz $u(x,t) = e^{it}Q(x)$. It can be proved that Q is unique, positive, spherically symmetric and very smooth. It also satisfies certain identities (Pohozaev's identities) that can be obtained by multiplying the elliptic equation by $\bar{u}$ and $x \cdot \nabla u$ and take the real part respectively. In particular the identities imply that $E(Q) = 0$. Weinstein discovered that the mass of the ground state is actually the best constant of the Gagliardo-Nirenberg inequality. More precisely by minimizing the functional

$$J(u) = \frac{\|\nabla u(t)\|_{L^2}^2 \|u\|_{L^2}^{\frac{4}{n}}}{\|u\|_{L_x^{2+\frac{4}{n}}}^{2+\frac{4}{n}}}$$

he showed that the best constant of

$$\frac{1}{2+\frac{4}{n}} \|u(t)\|_{L^{2+\frac{4}{n}}}^{2+\frac{4}{n}} \leq \frac{C}{2} \|\nabla u(t)\|_{L^2}^2 \|u(t)\|_{L^2}^{\frac{4}{n}},$$

is

$$C = \|Q\|_{L^2}^{-\frac{4}{n}}.$$

Thus using the calculations that we did before we have that

$$E(u_0) \geq \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 \left(1 - \frac{\|u_0\|_{L^2}^{\frac{4}{n}}}{\|Q\|_{L^2}^{\frac{4}{n}}} \right).$$

Thus if $\|u_0\|_{L^2} < \|Q\|_{L^2}$ we have a global solution. Moreover the condition is sharp in the sense that for any $\eta > \|Q\|_{L^2}$, there exists $u_0 \in H^1$ such that $\|u_0\|_{L^2} = \eta$, and $u(t)$ blows-up in finite time. To see this set

$$\gamma = \frac{\eta}{\|Q\|L^2} > 1,$$

and consider $u_0 = \gamma Q$. Then $\|u_0\|_{L^2} = \eta$ and

$$E(u_0) = \gamma^{2+\frac{4}{n}} E(Q) - \frac{\gamma^{2+\frac{4}{n}} - \gamma^2}{2} \|\nabla Q\|_{L^2}^2 = -\frac{\gamma^{2+\frac{4}{n}} - \gamma^2}{2} \|\nabla Q\|_{L^2}^2 < 0.$$

Since $u_0 = \gamma Q \in \Sigma$ and $E(u_0) < 0$, by the Zakharov-Glassey argument we have blow-up in finite time.

Remarks 1. As consequence of the pseudo-conformal transformation

$$u(x,t) \to (1-t)^{-\frac{n}{2}} e^{-\frac{i|x|^2}{4(1-t)}} u(\frac{t}{1-t}, \frac{x}{1-t}),$$

we actually have blow-up even for $\eta = \|Q\|_{L^2}$. We cite the book of Cazenave again for the details. It is interesting that the blow-up rate is $\frac{1}{t}$ and thus at least in the L^2-critical case the lower estimate we gave is not optimal for all blow-up solutions.

2. What about plain L^2 data? For $n \geq 3$ and radial data, a very recent result of Killip, Visan and Zhang proves global well-posedness for initial data with mass less than the mass of the ground state. In this case since the proof is not energy based, they also prove scattering. Their method was inspired by a series of very influential papers of Kenig and Merle who they address the energy critical focusing problem. For the defocusing energy critical problem which we won't address in these notes the problem is almost as hard as the mass critical. The breakthrough in this area came by Bourgain and developed later by Colliander, Keel, Staffilani, Takaoka and Tao.

3. For H^1 data there is also a minimal amount of concentration of the L^2 norm at the origin. We will present the spherically symmetric case which is due to Merle and Tsutsumi. The

radial assumption plus the treatment in dimension one was accomplished later by Nawa. We should mention that the concentration happens for all sequences of times tending to the maximal time $T_{\max}$. A byproduct of the Killip, Visan and Zhang result mentioned in remark 2 is that the solution concentrates the mass of the ground state but only along a single sequence of times.

Theorem 6. *Consider* (28) *with* $u_0 \in H^1(\mathbb{R}^n) \cap \{radial\}$ *in dimensions* $n \geq 2$. *Let* ρ *be any function* $(0,\infty) \to (0,\infty)$ *such that* $\lim_{s\downarrow 0} \rho(s) = \infty$ *and that* $\lim_{s\downarrow 0} s^{\frac{1}{2}}\rho(s) = 0$. *If* u *is the maximal solution of* (28) *and* $T_{\max} < \infty$ *then*

$$\liminf_{t\uparrow T_{\max}} \|u(t)\|_{L^2(\Omega_t)} \geq \|Q\|_{L^2},$$

where

$$\Omega_t = \left(x \in \mathbb{R}^n : \ |x| < |T_{\max} - t|^{\frac{1}{2}} \rho(T_{\max} - t) \right).$$

To prove the theorem we note that a result of Strauss states that a radial bounded sequence of functions in H^1 contains a subsequence that converges strongly in L^p for $2 < p < \frac{2n}{n-2}$. Now set

$$\lambda(t) = \frac{1}{\|\nabla u(t)\|_{L^2}}$$

so that

$$\lim_{t\uparrow T_{\max}} \lambda(t) = 0.$$

We claim that

$$\liminf_{t\uparrow T_{\max}} \|u(t)\|_{L^2(|x|<\lambda(t)\rho(T_{\max}-t))} \geq \|Q\|_{L^2}.$$

The result then follows since ρ is arbitrary and $\|\nabla u(t)\|_{L^2} \geq \frac{\delta}{(T_{\max}-t)^{\frac{1}{2}}}$.

We prove the claim by contradiction. Assume there exists $t_n \uparrow T_{\max}$ such that

$$\lim_{n\to\infty} \|u(t)\|_{L^2(|x|<\lambda(t_n)\rho(T_{\max}-t_n))} < \|Q\|_{L^2}.$$

Set

$$v_n(t) = \lambda(t)^{\frac{n}{2}} u(t_n, \lambda(t_n)x).$$

Clearly

$$\|v_{t_n}\|_{L^2} = 1,$$

$$\|\nabla v_n\|_{L^2} = 1, \tag{29}$$

$$E(v_n) = \lambda(t_n)^2 E(u(t_n)) = \lambda(t_n)^2 E(u_0).$$

In particular

$$E(v_n) = \frac{1}{2} - \frac{1}{2+\frac{4}{n}} \|v_n\|_{L^{2+\frac{4}{n}}}^{2+\frac{4}{n}},$$

and

$$\lim_{n\to\infty} E(v_n) = 0.$$

Thus

$$\lim_{n\to\infty} \|v_n\|_{L^{2+\frac{4}{n}}}^{2+\frac{4}{n}} \to \frac{2+\frac{4}{n}}{2} \neq 0. \tag{30}$$

Since v_n is a bounded H^1 sequence there exists a subsequence which we still denote by v_n that converges weakly to w in H^1 and strongly by Strauss result in $L^{2+\frac{4}{n}}$. By the properties of weak and strong limits and (30) we have that

$$E(w) \leq 0, \qquad w \neq 0.$$

By the sharp Gagliardo-Nirenberg inequality this means

$$\|w\|_{L^2} \geq \|Q\|_{L^2}.$$

Now given $M > 0$ we have that

$$\|w\|_{L^2(|x|<M)} = \lim_{n\to\infty} \|v_n\|_{L^2(|x|<M)} = \lim_{n\to\infty} \|u(t_n)\|_{L^2(|x|<M\lambda(t_n))}$$

$$\leq \liminf_{n\to\infty} \|u(t_n)\|_{L^2(|x|<\lambda(t_n)\rho(T_{\max}-t))}$$

since $\rho(s) \to \infty$ as $s \downarrow 0$. But since M was arbitrary we obtain

$$\liminf_{n\to\infty} \|u(t_n)\|_{L^2(|x|<\lambda(t_n)\rho(T_{\max}-t))} \geq \|w\|_{L^2} > \|Q\|_{L^2}$$

obtaining the contradiction.

4. For general L^2 data the best concentration result up to this date is due to Bourgain, where he proves that the L^2 concentrates a fixed amount of mass. The method fails to provide the mass of the ground state as the relevant threshold though.

5.3. **Quantum scattering in the energy space.** Consider the defocusing L^2-supercritical problem

$$\begin{cases} iu_t + \Delta u - |u|^{p-1}u = 0, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^n). \end{cases} \tag{31}$$

for any $1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}$.

We define the set of initial values u_0 which have a scattering state at $+\infty$ (by time reversibility all the statements are equivalent at $-\infty$):

$$\mathcal{R}_+ = (u_0 \in H^1 : T_{\max} = \infty, \quad u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t) \ exists\ ). \tag{32}$$

Now define the operator

$$U : \mathcal{R}_+ \to H^1.$$

This operator sends u_0 to the scattering state u_+. If this operator is injective then we can define the wave operator

$$\Omega_+ = U^{-1} : U(\mathcal{R}_+) \to \mathcal{R}_+$$

which sends the scattering state u_+ to u_0. Thus the first problem of scattering is the existence of wave operator:

• *Existence of wave operators. For each u_+ there exists unique $u_0 \in H^1$ such that $u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t)$.*

If the wave operator is also surjective we say that we have asymptotic completeness (thus in this case the wave operator is invertible):

• *Asymptotic completeness. For every $u_0 \in H^1$ there exists u_+ such that $u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t)$.*

Both of the above statements make rigorous the idea that we have scattering if as time goes to infinity the nonlinear solution of the NLS behaves like the solution of the linear equation.

Using the decay estimates of section 3 we can solve the scattering problem for every $p > 1 + \frac{4}{n}$. Well-defined wave operators for this range of p is easy and it is almost a byproduct of the local theory. But asymptotic completeness is hard. In dimensions $n \geq 3$ this is due to Ginibre and Velo and for $n = 1, 2$, the result is due to Nakanishi. Their arguments are clear but complicated. Using the interaction Morawetz estimates we can prove the scattering properties in two simple steps. To make the presentation clear we will only show the $n = 3$ case with the cubic nonlinearity. But keep in mind that the interaction

Morawetz estimates give global a priori control on quantities of the form

$$\|u\|_{L_t^q L_x^r} \leq C_{M(u_0),E(u_0)},$$

for certain q and r in all dimensions. It turns out that in the L^2-supercritical case this is enough to give scattering for any $p > 1 + \frac{4}{n}$ and n. Finally for completeness we also outline the wave operator question.

Theorem 7. *For every $u_+ \in H^1(\mathbb{R}^3)$ there exists unique $u_0 \in H^1(\mathbb{R}^3)$ such that the maximal solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ of $iu_t + \Delta u = |u|^2 u$, satisfies*

$$\lim_{t \to \infty} \|e^{-it\Delta} u(t) - u_+\| = 0.$$

Proof: For $u_+ \in H^1$ define the map

$$\mathcal{A}(u)(t) = e^{it\Delta} u_+ + i \int_t^\infty e^{i(t-s)\Delta} (|u|^2 u)(s) ds.$$

What is the motivation behind this map? Recall that

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} (|u|^2 u)(s) ds,$$

$$e^{-it\Delta} u(t) = u_0 - i \int_0^t e^{-is\Delta} (|u|^2 u)(s) ds. \tag{33}$$

If the problem scatters we have that $\lim_{t \to \infty} \|e^{-it\Delta} u(t) - u_+\| H^1 = 0$ and thus

$$u_+ = u_0 - i \int_0^\infty e^{-is\Delta} (|u|^2 u)(s) ds. \tag{34}$$

Now subtracting (34) from (33) we have that

$$u(t) = e^{it\Delta} u_+ + i \int_t^\infty e^{i(t-s)\Delta} (|u|^2 u)(s) ds.$$

By Strichartz estimates we have that

$$\|e^{it\Delta}u_+\|_{L_t^q W_x^{1,r}} \lesssim \|u_+\|_{H^1} < \infty.$$

By the monotone convergence Theorem there exists $T = T(u_+)$ very large such that for $q < \infty$ we have

$$\|e^{it\Delta}u_+\|_{L_t^q W_x^{1,r}} \lesssim \epsilon.$$

The trick here is to use the smallness assumption to iterate the map in the interval (T, ∞). But our local theory was performed in the norms

$$\|u\|_{\mathcal{S}^1(I\times\mathbb{R}^n)} = \|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} + \|\nabla u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)}$$

where

$$\|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} = \sup_{(q,r)-admissible} \|u\|_{L^q_{t\in I}L^r_x}.$$

But this norms contain L^∞. So momentarily we will go to the smaller space

$$X = L_t^5 L_x^5 \cap L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}.$$

But for this norm we know have that for latge T

$$\|e^{it\Delta}u_+\|_{X_{[T,\infty)}} \lesssim \epsilon.$$

Furthermore Strichartz estimates show that

$$\|\mathcal{A}(u)\|_{X_{[T,\infty)}} \lesssim \epsilon + \|u\|^3_{X_{[T,\infty)}}.$$

This is because by Sobolev embedding

$$\|f\|_{L_t^5 L_x^5} \lesssim \|f\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{30}{11}}}$$

where the pair $(5, \frac{30}{11})$ is admissible in H^1. Thus for T large enough we have that

$$\|u\|_{X_{[T,\infty)}} \lesssim \epsilon.$$

It remains to show that the solution is in $C([T,\infty); H^1(\mathbb{R}^3))$. But by Strichartz again and using any admissible pair we have

$$\|u\|_{L^q_{t\in[T,\infty)} W^{1,r}_x} \lesssim \|u_+\|_{H^1} + \|u\|^3_{X_{[T,\infty)}} \lesssim \|u_+\|_{H^1}.$$

In particular $\psi = u(T) \in H^1$ and we have a strong H^1 solution of the equation with initial data $u(T) = \psi$. But we know that the solutions of this equation are global and thus $u(0)$ is well-defined. Finally

$$e^{-it\Delta}u(t) - u_+ = i\int_t^\infty e^{-is\Delta}(|u|^2u)(s)ds,$$

$$\nabla(e^{-it\Delta}u(t) - u_+) = i\int_t^\infty e^{-is\Delta}(\nabla(|u|^2u))(s)ds,$$

$$\|e^{-it\Delta}u(t) - u_+\|_{H^1} \lesssim \|\nabla u\|_{L^{\frac{5}{3}}_{[t,\infty)}L^{10}_x}\|u\|^2_{L^5_{[t,\infty)}L^5_x} \lesssim \|u\|^2_{L^5_{[t,\infty)}L^5_x}$$

since $(\frac{5}{3}, 10)$ is admissible and thus $\|\nabla u\|_{L^{\frac{5}{3}}_{[t,\infty)}L^{10}_x} \leq C$. But for T large enough we have that $\|u\|_{X_{[T,\infty)}} \lesssim \epsilon$ and thus

$$\lim_{t\to\infty} \|e^{-it\Delta}u(t) - u_+\|_{H^1} = 0.$$

Therefore $u(0) = u_0 \in H^1$ satisfies the assumptions of the Theorem. We end with asymptotic completeness.

Theorem 8. *If $u_0 \in H^1(\mathbb{R}^3)$ and if $u \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ where u is the solution of $iu_t + \Delta u = |u|^2u$, then there exists u_+ such that*

$$\lim_{t\to\infty} \|e^{-it\Delta}u(t) - u_+\| = 0.$$

The proof is based on a simple proposition now that we have the interaction Morawetz estimates. This was the hardest part of the proof in the Ginibre and Velo papers.

Proposition 1. *Let u be a global H^1 solution of the cubic defocusing equation on $\mathbb{R}^3$. Then*

$$\|u\|_{\mathcal{S}^1(\mathbb{R}\times\mathbb{R}^3)} \leq C.$$

Proof: We know that $\|u\|_{L^4_t L^4_x} \leq C$ for energy solutions. Thus we can pick ϵ small to be determined later and a finite number of intervals $\{I_k\}_{k=1,2,\ldots,M}$, with $M < \infty$ such that

$$\|u\|_{L^4_{t\in I_k} L^4_x} \leq \epsilon$$

for all k. If we know apply the Strichartz estimates on I_k we obtain

$$\|u\|_{\mathcal{S}^1(I_k)} \lesssim \|u(0)\|_{H^1} + \|u\|^{2\alpha}_{L^4_{t\in I_k} L^4_x} \|u\|^{3-2\alpha}_{\mathcal{S}^1(I_k)}, \tag{35}$$

$$\|u\|_{\mathcal{S}^1(I_k)} \lesssim \|u(0)\|_{H^1} + \epsilon^{2\alpha} \|u\|^{3-2\alpha}_{\mathcal{S}^1(I_k)}.$$

We can pick ϵ so small such that

$$\|u\|_{\mathcal{S}^1(I_k)} \leq K.$$

Since the number of intervals are finite and the conclusion can be made for all $I_k's$ the Proposition follows.

Remarks. 1. Where do we use the condition $p > 1 + \frac{4}{n}$? This is a delicate matter. It is not hard to see that the interaction Morawetz estimates are global estimates of Strichartz type but are not L^2 scale invariant. If one inspects the right hand side of the interaction inequalities, a simple scaling argument shows that these are $H^{\frac{1}{4}}$ invariant estimates. Thus only in the case that $p > 1 + \frac{4}{n}$ we can take advantage of an non L^2 estimate such as $L^4_t L^4_x$. This is the heart of the matter in proving (35). In the case that $p = 1 + \frac{4}{n}$ we need to have a global L^2 Strichartz estimate like $L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}$ in dimensions 3. Estimates of this sort can never come from Morawetz estimates due to scaling.

2. Notice that the Proposition gives a global decay estimate for the nonlinear solution.

Let's finish the proof of asymptotic completeness. Note that

$$e^{-it\Delta}u(t) = u_0 - i\int_0^t e^{-is\Delta}(|u|^2u)(s)ds,$$

$$e^{-i\tau\Delta}u(\tau) = u_0 - i\int_0^\tau e^{-is\Delta}(|u|^2u)(s)ds.$$

Thus

$$\|e^{-it\Delta}u(t) - e^{-i\tau\Delta}u(\tau)\|_{H^1} = \|u(t) - e^{-i(t-\tau)\Delta}u(\tau)\|_{H^1} \lesssim \|u\|^3_{\mathcal{S}^1_{(t,\tau)}} \leq C$$

again by Strichartz estimates. Thus as $t, \tau \to \infty$ we have that

$$\|e^{-it\Delta}u(t) - e^{-i\tau\Delta}u(\tau)\|_{H^1} \to 0.$$

By completeness of H^1 there exists $u_+ \in H^1$ such that $e^{-it\Delta}u(t) \to u_+$ in H^1 as $t \to \infty$. In particular in H^1 we have

$$u_+ = u_0 - i\int_0^\infty e^{-is\Delta}(|u|^2u)(s)ds$$

and thus

$$\|e^{-it\Delta}u(t) - u_+\|_{H^1} \lesssim \|u\|^3_{\mathcal{S}^1_{(t,\infty)}}.$$

As $t \to \infty$ the conclusion follows.

Remark. What about energy scattering for $p \leq 1 + \frac{4}{n}$. The problem is completely open but more on this later. We have already said that in some sense scattering makes rigorous the idea that as time increases, for a defocusing problem, the nonlinearity $|u|^{p-1}u$ becomes negligible. From this it is almost common sense to conclude that the bigger the power of p the better chance the solution has to scatter. Thus the question: Is there any threshold p_0 with $1 < p_0 \leq 1 + \frac{4}{n}$ such that energy scattering does fail?. The answer is yes and $p_0 = 1 + \frac{2}{n}$. This is due to Strauss and Tsutsumi in higher dimensions and to Barab in dimension 1. More precisely using the pseudo-conformal conservation law and the estimates that we discuss later in the notes, they showed that for any $1 < p \leq p_0$, $U(-t)u(t)$ doesn't converge even in L^2. Thus the wave operators cannot exist in any reasonable set. Thus the problem remains open for

$$1 + \frac{2}{n} < p < 1 + \frac{4}{n}.$$

5.4. **Quantum scattering in the Σ space.** If we are willing to abandon the energy space can we improve scattering in the range $1 + \frac{2}{n} < p < 1 + \frac{4}{n}$? Recall that

$$\Sigma = H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n,\ |x|^2 dx).$$

We will not go into the details but a few comments can clarify the situation. Exactly like the energy case it is enough to prove that

$$\|u\|_{\mathcal{S}^1(\mathbb{R}\times\mathbb{R}^3)} \leq C.$$

How one can obtain this estimate for different values of p? First recall that for

$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 + \frac{8t^2}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx$$

we have that

$$K(t) - K(0) = \int_0^t \theta(s) ds,$$

where

$$\theta(t) = \frac{4t}{p+1} (4 - n(p-1)) \int_{\mathbb{R}^n} |u|^{p+1} dx.$$

Using this quantity and a simple analysis one can obtain the following Proposition:

Proposition 2. *Consider the defocusing NLS*

$$\begin{cases} iu_t + \Delta u = |u|^{p-1}u \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^n). \end{cases} \tag{36}$$

for any $1 < p < 1 + \frac{4}{n-2}$, $n \geq 3$ *(*$1 < p < \infty$ *for* $n = 1, 2$*). If in addition* $\|xu_0\|_{L^2} < \infty$ *and*

$$u \in C_t^0(\mathbb{R}; H^1(\mathbb{R}^n))$$

solves (36)*, then we have:*

i) If $p > 1 + \frac{4}{n}$ *then for any* $2 \leq r \leq \frac{2n}{n-2}$ *(*$2 \leq r \leq \infty$ *if* $n = 1$, $2 \leq r < \infty$ *if* $n = 2$*)*

$$\|u(t)\|_{L^r} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{r})}$$

for all $t \in \mathbb{R}^n$*.*

ii) If $p < 1 + \frac{4}{n}$ *then for any* $2 \leq r \leq \frac{2n}{n-2}$ *(*$2 \leq r \leq \infty$ *if* $n = 1$, $2 \leq r < \infty$ *if* $n = 2$*)*

$$\|u(t)\|_{L^r} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{r})(1-\theta(r))}$$

where

$$\theta(r) = \begin{cases} 0 & \text{if } 2 \leq r \leq p+1 \\ \frac{[r-(p+1)][4-n(p-1)]}{(r-2)[(n+2)-p(n-1)]} & \text{if } r > p+1. \end{cases}$$

Remarks. 1. Notice that for $p \geq 1 + \frac{4}{n}$ the decay is as strong as the linear equation. Recall here the basic $L^1 - L^\infty$ estimate of the linear problem and its interpolation with Plancherel.

2. Using these estimates and the standard theory we have developed one can prove that global solutions defined in the Σ space obey

$$\|u\|_{\mathcal{S}^1(\mathbb{R}\times\mathbb{R}^3)} \leq C,$$

for any

$$1+\frac{2-n+\sqrt{n^2+12n+4}}{2n}<p<1+\frac{4}{n-2}.$$

Existence of Wave operators and asymptotic completeness follows easily. Of course

$$1+\frac{2}{n}<1+\frac{2-n+\sqrt{n^2+12n+4}}{2n}<1+\frac{4}{n}.$$

3. The existence of the wave operators can go below the Strauss exponent in all dimensions and actually one can cover the full range $p>1+\frac{2}{n}$. The subject is rather technical and we refer to Cazenave's book for more details.

6. Blow-up, global well-posedness and scattering below $H^1(\mathbb{R}^n)$.

Now consider the defocusing NLS. Start with data $u_0\in H^s$, with $s<1$. We have seen that in order to have global solutions an a priori estimate of the form

$$\sup_{t\in\mathbb{R}}\|u(t)\|_{H^s}\leq C_{\|u_0\|_{H^s}}$$

will suffice. Absent a conservation law at this level it is not realistic to expect such a bound. On the other hand as we said we can relax the growth of the solutions in the Sobolev norms and still being able to prove global well-posedness. For example if we knew that our solutions are bounded only polynomially in time

$$\sup_{t\in[0,T]}\|u(t)\|_{H^s}\leq C_{\|u_0\|_{H^s}}(1+T)^{\rho},$$

then gwp would follow . In this case for each fixed large T we have an a priori bound that we can use to iterate the local solutions. Where can we find such a bound though? We present the main ideas of the so called I-method which was introduced by Colliander, Keel, Staffilani, Takaoka and Tao. The method was inspired by a series of papers of Bourgain addressing gwp for solutions with infinite energy.

Their method is based on studying a smooth out version of the NLS using a new operator that we will call the I-operator. First introduce a radial C^∞, monotone multiplier, taking values in $[0,1]$, where:

$$m(\xi) := \begin{cases} 1 & \text{if } |\xi| < N \\ (\frac{|\xi|}{N})^{s-1} & \text{if } |\xi| > 2N, \end{cases}$$

and we define $I : H^s \to H^1$ by

$$\widehat{Iu}(\xi) = m(\xi)\widehat{u}(\xi).$$

The operator I is smoothing of order $1-s$ and the following estimate holds

$$\|u\|_{H^s} \lesssim \|Iu\|_{H^1} \lesssim N^{1-s}\|u\|_{H^s}.$$

Thus the operator acts as the identity for low frequencies but sends H^s solutions to H^1. With the aid of this operator, the following functional now makes sense (we call it the **modified energy functional**)

$$E(Iu)(t) = \frac{1}{2}\int |\nabla Iu(t)|^2 dx + \frac{1}{p+1}\int |Iu(t)|^{p+1} dx = E(Iu_0).$$

Of course this quantity is not anymore conserved in time since the smooth out version of the solution Iu doesn't satisfy the equation. It rather satisfies formally the following equation

$$iIu_t + \Delta Iu = I(|u|^{p-1}u) =$$

$$|Iu|^{p-1}Iu + I(|u|^{p-1}u) - |Iu|^{p-1}Iu = |Iu|^{p-1}Iu + [I,F](u),$$

where F is the nonlinearity. The method will be successful if somehow this commutator is small as the problem evolves. If we now differentiate the modified energy in time, use the new equation for Iu and then use the fundamental theorem of calculus we obtain that

$$E(u)(T) - E(u)(0) = \int_0^T \frac{\partial}{\partial t} E(Iu(t))\ dt =$$

$$\Re \int_0^T \int_{\mathbb{R}^n} \overline{Iu_t}\left(-\Delta Iu + F(Iu)\right)\ dxdt = \Re \int_0^T \int_{\mathbb{R}^n} \overline{Iu_t}\left(F(Iu) - IF(u)\right)\ dxdt.$$

Notice that by Sobolev embedding for any $p < 1 + \frac{4}{n-2}$ we have that

$$\|Iu\|_{L^{p+1}} \lesssim \|Iu\|_{H^1}$$

and thus

$$E(Iu) \lesssim \|Iu\|_{H^1}^2.$$

The first step is to prove a modified lwp theory for the Iu which is a solution to

$$iIu_t + \Delta Iu = I(|u|^{p-1}u).$$

This is usually happens as we have shown in section 2 using the norms

$$\|Iu\|_{\mathcal{S}^1(I\times\mathbb{R}^n)} = \|Iu\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} + \|\nabla Iu\|_{\mathcal{S}^0(I\times\mathbb{R}^n)}$$

where

$$\|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} = \sup_{(q,r)-admissible} \|u\|_{L^q_{t\in I}L^r_x}.$$

Thus our solution Iu is well-defined on an interval $[0,\delta]$ where

$$\delta \sim \frac{1}{\|Iu_0\|_{H^1}^{\gamma}}.$$

On that interval we obtain the bound (through Duhamel)

$$\|Iu\|_{\mathcal{S}^1([0,\delta]\times\mathbb{R}^n)} \lesssim \|Iu_0\|_{H^1}.$$

We can always use scaling to make $\|Iu_0\|_{H^1} = O(1)$ but this is not important for the method. We can apply these techniques to systems of equations like the Zakharov system where the scaling of the wave equation is not consistent with the scaling of the Schrödinger equation.

Now assume that the error term in the modified functional is small on the same interval $[0, \delta]$. How small? We measure smallness by requiring the error term of the modified functional to decay with respect to the large parameter N. We will prove this estimate later but for the moment let's assume that we have it. Thus assume that

$$|\int_0^T \int_{\mathbb{R}^n} \overline{Iu_t}\, (F(Iu) - IF(u))\ dxdt| \lesssim N^{-\alpha} \|Iu\|^{\beta}_{\mathcal{S}^1([0,\delta]\times\mathbb{R}^n)}$$

for some $\alpha,\ \beta > 1$. Now for fix T divide the interval $[0, T]$ into

$$L \sim \frac{T}{\delta} \sim N^{\gamma(1-s)}$$

sub-intervals of length of order δ in which lwp and the modified energy error bound holds for the solution. We have that

$$\|\nabla Iu(\delta)\|^2_{L^2} \lesssim E(Iu)(\delta) \lesssim E(Iu)(0) + |\int_0^T \int_{\mathbb{R}^n} I\bar{u}_t\, (F(Iu) - IF(u))\ dxdt|$$

$$\lesssim \|\nabla Iu(0)\|^2_{L^2} + N^{-\alpha} \|Iu\|^{\beta}_{\mathcal{S}^1([0,\delta]\times\mathbb{R}^n)} \lesssim \|Iu(0)\|^2_{H^1} + N^{-\alpha}\|Iu(0)\|^{\beta}_{H^1}.$$

Recall that

$$\|Iu_0\|_{H^1} \lesssim N^{1-s}\|u_0\|_{H^s}$$

and thus

$$\|\nabla Iu(\delta)\|^2_{L^2} \lesssim N^{2(1-s)} + N^{-\alpha} N^{\beta(1-s)}.$$

Iterating the result over the whole interval $[0, T]$ we obtain

$$\|Iu(T)\|^2_{H^1} \lesssim N^{2(1-s)} + N^{-\alpha} N^{\beta(1-s)} L.$$

Notice that the L^2 part of the Sobolev norm doesn't grow since

$$\|Iu\|_{L^2} \leq \|u\|_{L^2} = \|u_0\|_{L^2} \leq C.$$

All in all we have

$$\|Iu(T)\|_{H^1}^2 \lesssim N^{2(1-s)} + TN^{-\alpha}N^{(\beta+\gamma)(1-s)}.$$

Now pick N sufficiently large such that

$$N^{2(1-s)} \sim TN^{(\beta+\gamma)(1-s)-\alpha}.$$

This process identifies

$$T \sim N^{k(s)}$$

with $k(s) > 0$, while simultaneously we retain that

$$\|u(T)\|_{H^s} \leq \|Iu(T)\|_{H^1} \lesssim N^{1-s} \lesssim (1+T)^{\frac{1-s}{k(s)}}.$$

The requirement that $k(s) > 0$ is responsible for the existence of a regularity index $1 > s_0 > 0$ such that for any $s > s_0$ we obtain the a priori bound

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq C_{\|u_0\|_{H^s}}(1+T)^{\rho}.$$

How can we improve the method? Since β is fixed and depends on the nonlinearity we can either improve the regularity by having a smaller γ or having a bigger α. Both methods have been applied in the past. Let's consider each separately:

–**Better decay of the modified energy implies larger γ.** The strategy here is straightforward. Better dispersive estimates will give a better exponent α. Since most of the dispersive estimates are known to be sharp there is little room for improvement here no matter in how many pieces we decompose the solutions. Another approach that has been employed in the past is to consider higher correction terms for the energy. This depends on the problem and it is usually restricted in few equations.

—**More efficient local modified theory implies smaller γ.** It turns out that a "better" (in the previous sense) modified theory can be built if one has global a priori estimates of the form

$$\|u\|_{K_t^q L_x^r} \leq C.$$

We have seen that these estimates are a by product of the interaction Morawetz estimate. But there is a catch!. For simplicity consider the case $n = 2$ where the estimate $L_t^4 L_x^8$ has been obtained. Moreover consider the L^2-critical problem with $p \geq 1 + \frac{4}{2} = 3$. The problem is lwp for any $s > 1 - \frac{2}{p-1}$ with the time of the local existence depending on the norm of the initial data. Recall the estimate

$$\|u\|^2_{L_t^4 L_x^8} \lesssim \|u(t)\|_{L^2} \|u(t)\|_{\dot{H}^{\frac{1}{2}}}.$$

If we are looking for proving gwp for $1 > s \geq \frac{1}{2}$, then the right hand side of the interaction estimate is well-defined and one can built an iteration that lowers the γ. This approach has been used by Fang and Grillakis for $p = 3$. Moreover in case that $p > 3$, an integer, one can prove global well-posedness and scattering below H^1.

But what about if $s < \frac{1}{2}$. Then we have also to smooth out the interaction Morawetz estimate with Iu. So consider the initial value problem

$$\begin{cases} iIu_t + \Delta Iu - I(|u|^2 u) = 0 \quad & x \in \mathbb{R}^2, \quad t \in \mathbb{R} \\ Iu(x,0) = Iu_0(x) \in H^1(\mathbb{R}^2). \end{cases} \tag{37}$$

The aim is to prove the following "almost Morawetz" estimate

$$\|Iu\|^4_{L^4_T L^4_x} \lesssim T^{\frac{1}{2}} \sup_{t\in[0,T]} \|Iu(t)\|^2_{\dot{H}^1} \|Iu_0\|^2_{L^2} + Error. \tag{38}$$

Recall that differentiation of the Morawetz action yields 4 terms. Three of them are coming from the linear equation and the last one P_4 (look at section 3) comes from the nonlinearity. But keep in mind that the I operator commutes with the linear equation. Now for the tensor product $u(x,t) = u(x_1,t)u(x_2,t)$ we define

$$M^I_a(t) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \nabla a \cdot \Im(I\bar{u}\nabla Iu)\ dx.$$

The linear part of the proof can continue as before. But the nonlinearity will produce again a term that contains the error

$$\big(I(|u(x_1)|^2u(x_1)) - |Iu(x_1)|^2 Iu(x_1)\big) \times |Iu(x_2)|^2 +$$

$$\big(I(|u(x_2)|^2u(x_2)) - |Iu(x_2)|^2 Iu(x_2)\big) \times |Iu(x_1)|^2.$$

Using Strichartz estimates we show that if $J = [a,b]$ and

$$\|Iu\|^4_{L^4_t L^4_x(J\times\mathbb{R}^2)} < \mu,$$

where μ is a small universal constant, then

$$\sup_{(q,r)\quad admissible} \|(1+\nabla)Iu^\lambda\|_{L^q_t L^r_x(J\times\mathbb{R}^2)} \lesssim \|Iu^\lambda(a)\|_{H^1} \lesssim 1.$$

This is what improves the γ. Notice that the $L^4_t L^4_x$ estimate is admissible. We then continue as before by iterating both the error of the Morawetz and the error of the energy. Notice that the form of both errors are similar.

We would like to add a few comments on how one proves the decay estimate of the error terms. We consider the focusing L^2-critical NLS in dimension 2. Recall that

$$E(Iu(\delta)) - E(Iu(0)) = \Re \int_0^\delta \int_{\mathbb{R}^2} \overline{Iu_t} \left(|Iu|^2 Iu - I(|u|^2 u) \right) dxdt.$$

If we employ the Fourier transform on the x variable we have

$$E(Iu(\delta)) - E(Iu(0)) = \int_0^\delta \int_{\sum_{j=1}^4 \xi_j = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \widehat{\overline{Iu_t}}(\xi_1) \widehat{\overline{Iu}}(\xi_2) \widehat{\overline{Iu}}(\xi_3) \widehat{\overline{Iu}}(\xi_4) \right).$$

Notice that low frequencies of the solutions are in the energy space and we do not anticipate problems. Moreover if all frequencies are much smaller than the parameter N, then by definition $m(\xi) = 1$,

$$1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} = 0,$$

and there is nothing to prove. On the other hand very high frequencies are easier to handle and they give more decay. The higher frequency part of the solutions decay more in less regular norms and the lower local theory does the trick. It is somewhat tricky to deal with the *low-high* interactions. But this where we use the smoothness of the multiplier m. As an example suppose that

$$|\xi_2| \gg |\xi_3| \sim |\xi_4|, \qquad |\xi_3|, \ |\xi_4| \leq N.$$

Then

$$1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} = 1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)} = \frac{m(\xi_2) - m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)}.$$

By the mean value theorem we know that there exists $\xi \in (\xi_2, \ \xi_2+\xi_3+\xi_4)$ such that

$$|m(\xi_2) - m(\xi_2+\xi_3+\xi_4)| \lesssim |\xi_3+\xi_4| \ |\nabla m(\xi)|.$$

But $|\xi| \sim |\xi_2|$ and thus

$$|\frac{m(\xi_2) - m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)}| \lesssim \frac{|\xi_3|}{|\xi_2|}.$$

So we gain even in this case. This is the heart of the matter in the I method. The rest is Hölder's inequality although the selection of the norms that appear on the right hand side of the decay inequality is as important. The norms of course must agree with the norms of the modified local well-posedness theory. Traditionally this has been done in the mixed Lebesgue spaces or the $X^{s,b}$ spaces of Bourgain.

Notice that we use the fact that $|\xi_2| \gtrsim N$ as the highest frequency and the fact that

$$|\nabla m(\xi)| \sim \frac{|\xi|^{s-2}}{N^{s-1}}$$

and thus

$$|\frac{\nabla m(\xi)}{m(\xi)}| \sim \frac{1}{|\xi|}.$$