1. The energy method

Consider the KdV equation:

$$u_t + u_{xxx} + uu_x = 0$$

with initial data $u(0, x) = u_0(x) \in H^s(\mathbb{R})$ with $s$ being a sufficiently large integer. In this section, we say $u$ is a classical solution of KdV in $H^s$ if

$$u \in C([-\delta, \delta]; H^s) \cap C^1([-\delta, \delta]; H^{s-3})$$

and if $u$ satisfies KdV for each $x$ and $t$.

We start with the following energy inequality: if $u$ is a smooth solution of KdV, then there exists $T_0 = T_0(\|u_0\|_{H^s})$ such that on $[0, T_0]$,

$$\|u\|_{H^s} \leq 2\|u_0\|_{H^s}.$$ 

Indeed,

$$\partial_t \|\partial_x^s u\|_{L^2}^2 = 2 \int \partial_x^s u_t \partial_x^s u dx$$

$$= -2 \int \partial_x^{s+3} u \partial_x^s u dx - 2 \int \partial_x^s (uu_x) \partial_x^s u dx.$$
The first term is zero, the highest order contribution of the second term is

\[-2 \int u \partial_x^{s+1} u \partial_x^s u dx = \int u_x (\partial_x^s u)^2 dx.\]

Thus, we obtain for \(s > 3/2\)

\[\partial_t \|u\|_{H^s}^2 \lesssim \|u_x\|_{L^\infty} \|u\|_{H^s}^2 \lesssim \|u\|_{H^s}^3.\]

Integrating in time implies that

\[\|u(T)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 + \int_0^T \|u(\tau)\|_{H^s}^2 d\tau.\]

Let \(T_0 = \inf \{T : \|u\|_{H^s} \geq 2\|u_0\|_{H^s}\}\). Then on \([0, T_0]\), we have

\[\|u(T)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 + 8T_0 \|u_0\|_{H^s}^3.\]

This implies that \(T_0 \geq (8\|u_0\|_{H^s})^{-1}\).

**Remark.** We also note that the above argument implies via Gronwall’s inequality that

\[\|u\|_{H^s} \leq C\|u_0\|_{H^s} \exp \left( C \int_0^t \|u_x\|_{L^\infty} dt' \right),\]

where \(C\) is an absolute constant. The advantage of this inequality is that the time that it is valid depends only on the lower index Sobolev norm (\(H^2\) is enough, available by the energy inequality) of the initial data, whereas the a priori energy bound depends on the \(H^s\) norm.

To prove the existence and uniqueness of solutions we use parabolic regularization and consider

\[u_t + \epsilon u_{xxxx} + u_{xxx} + uu_x = 0.\]
The energy inequality above remains intact since the contribution of the parabolic term is negative. Local well-posedness of this equation is proved by running a contraction argument in the space

\[ X_T = \{ u \in C([0, T]; H^s) : u(0, x) = u_0(x), \sup_{t \in [0,T]} \| u(t) \|_{H^s} \leq 2 \| u_0 \|_{H^s} \}, \]

for the operator

\[ \Gamma u = e^{-\epsilon \partial_x^4} u_0 + \int_0^t e^{-\epsilon (t-\tau) \partial_x^4} (u_{xxx} + uu_x) dx. \]

We have the following inequalities for the heat kernel:

\[ \| e^{-\epsilon t \partial_x^4} u \|_{H^s} \leq \| u \|_{H^s} \]
\[ \| e^{-\epsilon t \partial_x^4 \partial_x^3 u} \|_{H^s} \lesssim \frac{1}{\epsilon^{3/4} t^{3/4}} \| u \|_{H^s} \]
\[ \| e^{-\epsilon t \partial_x^4 (uu_x)} \|_{H^s} \lesssim \frac{1}{\epsilon^{1/4} t^{1/4}} \| u \|^2_{H^s}. \]

The first one follows from the boundedness of the multiplier \( e^{-\epsilon t \xi^4} \). The second follows by the inequality

\[ |\xi^3 e^{-\epsilon t \xi^4}| \lesssim \frac{1}{\epsilon^{3/4} t^{3/4}}. \]

The third one follows similarly using the algebra property of Sobolev spaces.

Using these inequalities for \( \Gamma \), we obtain

\[ \| \Gamma u(T) \|_{H^s} \leq \| u_0 \|_{H^s} + \int_0^T \frac{1}{\epsilon^{3/4} (T - \tau)^{3/4}} \| u \|_{H^s} d\tau + \int_0^T \frac{1}{\epsilon^{1/4} (T - \tau)^{1/4}} \| u \|^2_{H^s} d\tau \]
\[ \leq \| u_0 \|_{H^s} + C \epsilon^{-3/4} T^{1/4} \| u_0 \|_{H^s} + C \epsilon^{-1/4} T^{3/4} \| u_0 \|^2_{H^s} \]
\[ \leq 2 \| u_0 \|_{H^s}, \]
if \( T \leq T_1(\epsilon, \|u_0\|_{H^s}) \). Therefore, iterating this local result using the energy inequality we obtain a solution, \( u^\epsilon \), valid in the time interval \([0, T_0]\). Also note that, using the equation, we have \( u^\epsilon \in C^1([0, T_0], H^{s-4}) \). From now on we will denote \( T_0 \) by \( T \).

Now we need to prove that \( u^\epsilon \) converges to a solution of KdV as \( \epsilon \) tends to zero. To do this we prove that \( u^\epsilon \) is Cauchy in \( C([0, T]; L^2) \). Take \( 0 < \epsilon < \epsilon' \) and consider the corresponding solutions. Using the equation for \( \epsilon \) and \( \epsilon' \), we have

\[
\partial_t \|u^\epsilon - u^{\epsilon'}\|_{L^2}^2 = -2\epsilon' \int (u^\epsilon - u^{\epsilon'}) \partial_x^4 (u^\epsilon - u^{\epsilon'}) - 2(\epsilon - \epsilon') \int (u^\epsilon - u^{\epsilon'}) \partial_x^4 u^\epsilon \\
- \int (u^\epsilon - u^{\epsilon'}) \partial_x^2 (u^\epsilon - u^{\epsilon'}) - \frac{1}{2} \int (u^\epsilon - u^{\epsilon'}) \partial_x [(u^\epsilon - u^{\epsilon'})(u^\epsilon + u^{\epsilon'})] \\
= -2\epsilon' \int (u^\epsilon - u^{\epsilon'}) \partial_x^4 (u^\epsilon - u^{\epsilon'}) - 2(\epsilon - \epsilon') \int (u^\epsilon - u^{\epsilon'}) \partial_x^4 u^\epsilon \\
- \frac{1}{4} \int (u^\epsilon - u^{\epsilon'})^2 \partial_x (u^\epsilon + u^{\epsilon'}) \\
\leq -2(\epsilon - \epsilon') \int (u^\epsilon - u^{\epsilon'}) \partial_x^4 u^\epsilon - \frac{1}{4} \int (u^\epsilon - u^{\epsilon'})^2 \partial_x (u^\epsilon + u^{\epsilon'}). 
\]

The second equality follows by noting that the third integral is zero. The last inequality follows by the inequality

\[
-2\epsilon' \int (u^\epsilon - u^{\epsilon'}) \partial_x^4 (u^\epsilon - u^{\epsilon'}) = -2\epsilon' \int [\partial_x^2 (u^\epsilon - u^{\epsilon'})]^2 \leq 0.
\]

We estimate the remaining terms by Cauchy Schwarz and Sobolev embedding (for \( \|\partial_x u\|_{L^\infty} \)) to obtain

\[
\partial_t \|u^\epsilon - u^{\epsilon'}\|_{L^2}^2 \lesssim |\epsilon - \epsilon'| \|u^\epsilon - u^{\epsilon'}\|_{L^2} \|u^\epsilon\|_{H^4} + \|u^\epsilon - u^{\epsilon'}\|_{L^2}^2 (\|u^\epsilon\|_{H^4} + \|u^{\epsilon'}\|_{H^4}).
\]

This implies that

\[
\partial_t \|u^\epsilon - u^{\epsilon'}\|_{L^2} \lesssim |\epsilon - \epsilon'| \|u^\epsilon\|_{H^4} + \|u^\epsilon - u^{\epsilon'}\|_{L^2} (\|u^\epsilon\|_{H^4} + \|u^{\epsilon'}\|_{H^4}).
\]
Integrating from 0 to $T$ for small $T$ and using the apriori bound $\|u^\epsilon\|_{H^4} \lesssim \|u_0\|_{H^4}$, we obtain
\[
\sup_{[0,T]} \|u^\epsilon - u'^\epsilon\|_{L^2} \lesssim |\epsilon - \epsilon'|.
\]
Therefore $u^\epsilon$ is Cauchy in $C([0,T]; L^2)$. By interpolation with $L^\infty H^s$, $u^\epsilon$ is also Cauchy in $C([0,T]; H^r)$ for any $r \in [0,s)$. Moreover using the equation, we conclude that $\partial_t u^\epsilon$ is Cauchy in $C([0,T]; H^{r-4})$. Therefore, the limiting function $u \in C([0,T]; H^r) \cap C^1([0,T]; H^{r-4})$ solves KdV (by taking pointwise limits). Also note that since $u^\epsilon$ is bounded in $H^s$ and converges to $u$ in $L^2$, $u \in L^\infty([0,T]; H^s)$.

We now continue with uniqueness. Consider KdV with initial data $u_0$ and $v_0$ in $H^s$. Let $u$ and $v$ be the corresponding solutions valid in a common time interval $[0,T]$. By using the equation as above we obtain
\[
\partial_t \|u - v\|_{L^2}^2 = -2 \int (u - v) \partial_x^3 (u - v) - \int (u - v) \partial_x (u^2 - v^2) \lesssim \|u - v\|_{L^2}^2 (\|u\|_{H^s} + \|v\|_{H^s}).
\]
Therefore by Gronwall we obtain
\[
\|u - v\|_{L^2} \lesssim \|u_0 - v_0\|_{L^2}
\]
on $[0,T]$. This implies uniqueness.

It remains to prove the continuous dependence on initial data and that $u \in C([0,T]; H^s)$ (this implies that $u \in C^1([0,T]; H^{s-3})$). To do this regularize the initial data as follows:
\[
u^\delta_0 := u_0 * \varphi_\delta.
\]
Here $\varphi$ is a Schwartz function with mean 1 satisfying $\partial_k^k \hat{\varphi}(0) = 0$ for all $k > 0$ (take $\hat{\varphi}$ constant 1 in a neighborhood of the origin), and $\varphi_\delta(x) = \frac{1}{\delta} \varphi(\frac{x}{\delta})$. Since $\varphi_\delta$ is an approximate
identity, \( u_0^\delta \) converges to \( u_0 \) in \( H^s \). Note also that \( \| u_0^\delta \|_{H^s} \lesssim \| u_0 \|_{H^s} \) where the implicit constant is independent of \( \delta \). Let \( u^\delta \) be the solution with initial data \( u_0^\delta \) on \([0, T]\) (coming from the parabolic regularization).

We will need the following lemma

**Lemma 1.** Consider the solutions \( u^\delta \) constructed above on \([0, T]\). We claim that

i) \( \| u^\delta \|_{H^{s+1}} \lesssim 1/\delta \),

ii) \( u^\delta \) converges to \( u \) in \( L^\infty_{[0,T]} H^s \),

iii) \( \| u^\delta - v^\delta \|_{H^s} \lesssim e^{CT/\delta} \| u_0 - v_0 \|_{H^s} \),

iv) Assume that \( u_{n,0} \) converges to \( u_0 \) in \( H^s \). Let \( u_n \) and \( u_n^\delta \) be the solutions corresponding to the initial data \( u_{n,0} \) and \( u_{n,0}^\delta \), respectively. Then

\[
\sup_n \| u_n^\delta - u_n \|_{H^s} \to 0 \quad \text{as} \ \delta \to 0.
\]

The bound i) and interpolation imply that

\[ u^\delta \in C([0, T]; L^2) \cap L^\infty_{[0,T]} H^{s+1} \subset C([0, T]; H^s). \]

This implies that \( u \in C([0, T]; H^s) \), since by ii) \( u^\delta \) converges to \( u \) uniformly on \([0, T]\).

The lemma also implies continuous dependence on initial data as follows. Assume that \( u_{n,0} \) converges to \( u_0 \) in \( H^s \). Construct the regularized solutions as in the lemma. Using triangle inequality and iii), we have (for \( t \in [0, T] \))

\[
\| u - u_n \|_{H^s} \leq \| u - u^\delta \|_{H^s} + \| u^\delta - u_n^\delta \|_{H^s} + \| u_n^\delta - u_n \|_{H^s} \\
\lesssim \| u - u^\delta \|_{H^s} + e^{CT/\delta} \| u_0 - u_{n,0} \|_{H^s} + \sup_j \| u_j^\delta - u_j \|_{H^s}.
\]
Given \( \epsilon > 0 \), fix \( \delta_0 \) sufficiently small so that in light of ii) and iv) we have

\[
\| u - u_n \|_{H^s} \lesssim \epsilon + e^{CT/\delta_0} \| u_0 - u_{n,0} \|_{H^s} + \epsilon.
\]

Taking \( n \) to \( \infty \) finishes the proof. It remains to prove the lemma.

**Proof of Lemma 1.** i) We first recall that on \([0,T]\)

\[
\| u^\delta \|_{H^{s+1}} \lesssim \| u_0^\delta \|_{H^{s+1}} \exp \left( C \int_0^t \| u_{x}^\delta \|_{\infty} \, dt' \right) \lesssim \| u_0^\delta \|_{H^{s+1}} \exp \left( CT \| u_0 \|_{H^s} \right).
\]

Therefore, it suffices to prove i) at time 0. Indeed,

\[
\| u_0^\delta \|_{H^{s+1}} = \| \langle \xi \rangle \hat{\varphi}(\delta \xi) \langle \xi \rangle^s \hat{u}_0(\xi) \|_{L^2} \lesssim \| \langle \xi \rangle \hat{\varphi}(\delta \xi) \|_{L^\infty} \| u_0 \|_{H^s} \lesssim 1/\delta.
\]

ii) We first prove that for \( 0 < \delta' < \delta \), we have

\[
\| u^\delta - u^{\delta'} \|_{L^2} = o(\delta^s).
\]

By (1), it suffices to prove this at time zero. We have

\[
\| u_0^\delta - u_0^{\delta'} \|_{L^2}^2 = \int |\hat{\varphi}(\delta \xi) - \hat{\varphi}(\delta' \xi)|^2 \langle \xi \rangle^{2s} |\hat{u}_0(\xi)|^2 \langle \xi \rangle^{2s} d\xi.
\]

By Taylor expansion and the fact that derivatives of \( \hat{\varphi} \) vanishes at zero, we have

\[
\hat{\varphi}(\delta \xi) = 1 + O(\delta^s \xi^s \sup_{[0,\delta\xi]} |\partial^s \hat{\varphi}|).
\]

Thus, we have

\[
(2) \quad \| u_0^\delta - u_0^{\delta'} \|_{L^2}^2 \lesssim \delta^2 s \int \| \partial^s \hat{\varphi} \|_{L^\infty([0,\delta\xi])} |\hat{u}_0(\xi)|^2 \langle \xi \rangle^{2s} d\xi.
\]

Since \( \partial^s \hat{\varphi}(0) = 0 \), the statement follows from the dominated convergence theorem.
Interpolating this inequality with the bound i), we obtain \( \|u^δ - u^δ'\|_{H^s} = o(1) \) as \( δ, δ' \) go to zero. This implies that \( u^δ \) is a convergent sequence in \( L^∞([0,T]; H^s) \). By (1),

\[
\|u^δ - u\|_{L^2} \lesssim \|u^δ_0 - u_0\|_{L^2} \to 0
\]
as \( δ \to 0 \), therefore \( u \) is the limit of \( u^δ \) also in \( H^s \).

iii) Using the equation, we estimate

\[
\partial_t \|\partial_x^s (u^δ - v^δ)\|_{L^2}^2 = 2 \int \partial_x^s (u^δ - v^δ) \partial_x^s (u^δ - v^δ) dx
\]

\[
= -2 \int \partial_x^{s+3} (u^δ - v^δ) \partial_x^s (u^δ - v^δ) dx - \int \partial_x^{s+1} ((u^δ)^2 - (v^δ)^2) \partial_x^s (u^δ - v^δ) dx
\]

\[
\lesssim \|u^δ - v^δ\|_{H^s}^2 (\|u^δ\|_{H^{s+1}} + \|v^δ\|_{H^{s+1}}) \lesssim \frac{1}{δ} \|u^δ - v^δ\|_{H^s}^2.
\]
The first inequality follows since the first summand is zero and the second one can be estimated by considering the cases when \( s + 1 \) derivatives hit \( u^δ + v^δ \) and \( u^δ - v^δ \). The second inequality follows from i).

This implies iii) by Gronwall.

iv) Since \( \|u^δ_n - u_n\|_{H^s} = \lim_{δ' \to 0} \|u^δ_n - u^δ'_n\|_{H^s} \), it suffices to prove that

\[
\sup_n \|u^δ_n - u^δ'_n\|_{L^2} = o(δ^s).
\]
Interpolation with the bound i) yields the claim.

Using (1) and (2) it is enough to show that

\[
\sup_n \int \|\partial_x^s \hat{\varphi}\|_{L^∞([0,δξ])} |\bar{u}_{n,0}(ξ)|^2 \langle ξ \rangle^{2s} dξ = o(1).
\]

Indeed,
\[ \leq \int \|\partial^s \hat{\varphi}\|_{L^\infty([0,\delta\xi])} |\hat{u}_0|^2 \langle \xi \rangle^{2s} d\xi + \sup_n \int \|\partial^s \hat{\varphi}\|_{L^\infty([0,\delta\xi])} |\hat{u}_{n,0} - \hat{u}_0|^2 \langle \xi \rangle^{2s} d\xi. \]

The first integral goes to zero by Lebesgue dominated convergence theorem. For the second, given \( \epsilon > 0 \), choose \( N \) so that \( \|u_{n,0} - u_0\|_{H^s} < \epsilon \) for all \( n > N \), and estimate the first \( N \) terms by dominated convergence theorem.

\[ \square \]

It remains to prove that the solutions constructed above can be defined globally-in-time. It is a well-known fact in the literature that smooth solutions of the KdV satisfy infinitely many conservation laws. A sample includes the following:

\[ I_1(t) = \int u(x,t)dx = I_1(0) \]

\[ I_2(t) = \int u^2(x,t)dx = I_2(0) \]

\[ I_3(t) = \int \left( u_x^2 - \frac{1}{3}u^3(x) \right) dx = I_3(0) \]

which can be verified directly by taking the time derivative of the above quantities and show that \( \partial_t I_j = 0, j = 1, 2, 3 \). Each conservation law along with interpolation provides an a priori bound

\[ \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} \]

for \( s \) an integer. In general for a given \( H^s \) solution, to make sense of the time differentiation and then integration by parts, we need the solutions to live in a smoother than \( H^s \) space (\( H^{s+3} \) suffices for the KdV). We resolve this minor problem by considering smooth solutions as follows. Let \( u_0 \in H^s \) and as before construct \( u_0^\delta \) smooth such that \( u_0^\delta \to u_0 \) in \( H^s \). By
continuous dependence we know that $u^\delta \to u$ in $H^s$. Since $u^\delta$ is smooth it satisfies the a priori bound (3). But then

$$\|u\|_{H^s} \leq \|u^\delta - u\|_{H^s} + \|u^\delta\|_{H^s} \lesssim \|u^\delta - u\|_{H^s} + \|u^\delta_0\|_{H^s} \lesssim \|u_0\|_{H^s}$$

by taking $\delta \to 0$. Thus $u$ satisfies the a priori bound. But then we can iterate the local solution and reach any time interval $[0,T]$. To see this assume that we have solve the problem locally-in-time and obtain a solution on $[0,T_1]$, $T_1 = f(\|u_0\|_{H^s})$, where on the same interval the solution satisfies (3). Now solve KdV with initial data $u(T_1)$ and obtain a solution on $[T_1,T_2]$ with $T_2 - T_1 = f(\|u(T_1)\|_{H^s}) = cf(\|u_0\|_{H^s})$ by the a priori bound (3). By continuity we can glue the solution together and thus $u$ solves KdV on $[0,T_2]$ and on this interval now it satisfies (3). Then $T_3 - T_2 = f(\|u(T_2)\|_{H^s}) = cf(\|u_0\|_{H^s})$. We continue with a uniform time step to cover $[0,T]$.

**Remark:** Notice that we cannot iterate the a priori bound coming from the energy inequality. This is because, as the $\|u\|_{H^s}$ grows, going from one time interval to another, the time intervals shrinks. Thus it is possible that the sequence of times shrinks in such a way that it approaches a finite time limit and the process stops.

2. Kenig–Ponce–Vega method on $\mathbb{R}$

We start with estimates for the linear KdV

$$u_t + u_{xxx} = 0, \quad u(0,x) = u_0(x).$$

The solution $W(t)u_0$ is given by convolution with the Airy kernel

$$A_t(x) = \int e^{it\xi^3 + ix\xi} d\xi.$$
Note that \( \| W(t)u_0 \|_{H^s} = \| u_0 \|_{H^s} \) for all real \( s \). We have the following dispersive decay estimate:

**Lemma 2.** For \( \alpha \in [0, 1/2] \),

\[
D^\alpha A_t(x) = \int e^{it\xi^3 + i2\xi} |\xi|^\alpha d\xi
\]

satisfies the bound

\[
\| D^\alpha A_t \|_{L^\infty} \lesssim |t|^{-(\alpha+1)/3}.
\]

**Proof.** By the scaling relation

\[
|D^\alpha A_t(x)| = |t|^{-(\alpha+1)/3} |D^\alpha A_t(x/t^{1/3})|
\]

it suffices to prove that \( |D^\alpha A_1(x)| \) is a bounded function. Since the estimate is trivial on the interval \( |\xi| < 2 \), it suffices to consider the integral

\[
\int e^{it\xi^3 + i2\xi} \psi(\xi) |\xi|^\alpha d\xi,
\]

where \( \psi \) is a \( C^\infty \) function satisfying \( \psi(\xi) = 0 \) for \( |\xi| < 1 \) and \( \psi(\xi) = 1 \) for \( |\xi| \geq 2 \). If \( x > -1 \), the estimate follows from one integration by parts by writing

\[
e^{i\xi^3 + i2} = \frac{1}{3i\xi^2 + ix}partial_\xi e^{i\xi^3 + i2}
\]

since then

\[
\frac{|\xi|^\alpha}{|3\xi^2 + x|} \lesssim 1.
\]

For \( x < -1 \) divide the integral into two pieces using smooth cutoffs \( \psi_1 \) and \( \psi_2 \) (\( \psi_1 + \psi_2 = 1 \)) where \( \psi_1 \) is supported on the set

\[
A = \{ \xi : |3\xi^2 + x| < |x|/2 \}.
\]
and $\psi_2$ is supported on

$$B = \{ \xi : |3\xi^2 + x| > |x|/3 \}.$$ 

To estimate the contribution of $\psi_2$ integrate by parts to obtain the bound

$$\int | d(3\xi^2 + x)| d\xi.$$ 

Since on the set $B$, $|3\xi^2 - x| \leq |3\xi^2 + x| + 2|x| \lesssim |3\xi^2 + x|$, we obtain $|3\xi^2 + x| \gtrsim |x|\gtrsim \langle \xi \rangle^2$. Also note that the derivative inside the integral can change sign at most a finite number of times. Therefore by fundamental theorem of calculus it suffices to see that

$$\left| \frac{|\xi|^\alpha \psi(\xi)\psi_2(\xi)}{3\xi^2 + x} \right| \lesssim \frac{|\xi|^\alpha}{\langle \xi \rangle^2} \lesssim 1.$$ 

To estimate the contribution of $\psi_1$, note that when $\xi \in A$, $\xi^2 \approx |x|$, and hence the second derivative of the phase, $\xi^3 + \xi x$, is $\gtrsim |x|^{1/2}$. Therefore by Van der Corput Lemma we estimate the contribution of $\psi_1$ by

$$|x|^{-1/4} \left( ||| |\xi|^\alpha \psi_1(\xi)\psi(\xi)||_{L^\infty} + \left\| \frac{d}{d\xi} \left( |\xi|^\alpha \psi_1(\xi)\psi(\xi) \right) \right\|_{L^1} \right).$$ 

Since the derivative changes sign at most finitely many times the two norms have the same contribution $\lesssim |x|^{\alpha/2}$. Therefore for $\alpha \in [0, 1/2]$, we obtain a uniform bound. $\square$

**Theorem 1.** *(Dispersive decay estimate)* For any $\theta \in [0, 1]$ and $\alpha \in [0, 1/2]$, we have

$$\| D^{\alpha \theta} W_t u_0 \|_{L^2/(1-\theta)} \lesssim |t|^{-\theta(\alpha+1)/3} \| u_0 \|_{L^2/(1+\theta)}$$

**Proof.** Writing

$$W(t)u_0 = A_t * u_0 = \int e^{it\xi^3 + i(x-y)\xi} u_0(y) d\xi dy,$$

the lemma above implies that

$$\| D^{\alpha} W_t u_0 \|_{L^\infty} \lesssim |t|^{-(\alpha+1)/3} \| u_0 \|_{L^1}.$$
The theorem will follow from complex interpolation of this bound with the $L^2$ conservation bound. To do this consider the analytic family of operators

$$D^z W(t) u_0 = D^z A_t * u_0$$

where $z = \alpha + i\beta$, $\alpha \in [0, 1/2]$, $\beta \in \mathbb{R}$. Since $D^{i\beta}$ is unitary, the operator is uniformly bounded in $L^2$ for $\alpha = 0$. Repeating the proof of the lemma above with $|\xi|^{\alpha + i\beta}$ instead of $|\xi|^\alpha$ gives

$$\|D^{\alpha + i\beta} W(t) u_0\|_{L^\infty} \lesssim \langle \beta \rangle |t|^{-(\alpha + 1)/3} \|u_0\|_{L^1}.$$ 

Therefore complex interpolation between the lines $\Re(z) = 0$ and $\Re(z) = \alpha$ yield the theorem.

**Theorem 2.** (Strichartz estimates) For any $\theta \in [0, 1]$ and $\alpha \in [0, 1/2]$, we have

(4) $$\|D^{\alpha \theta/2} W_t u_0\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2}$$

(5) $$\left\| \int_0^t D^{\alpha \theta} W_{t-\tau} g(\cdot, \tau) d\tau \right\|_{L^q_t L^r_x} \lesssim \|g\|_{L^{q'}_t L^{r'}_x},$$

where $(q, r) = (6/(\theta(\alpha + 1)), 2/(1 - \theta))$.

**Proof.** As usual by the $TT^*$ argument (with $T = D^{\alpha \theta/2} W_t$) (4) follows from the bound

$$\left\| \int_\mathbb{R} D^{\alpha \theta} W_{t-\tau} g(\cdot, \tau) d\tau \right\|_{L^q_t L^r_x} \leq \left\| \int_\mathbb{R} D^{\alpha \theta} W_{t-\tau} g(\cdot, \tau) d\tau \right\|_{L^q_t} \lesssim \left\| \int_\mathbb{R} |t - \tau|^{-\theta(\alpha + 1)/3} \|g\|_{L^{r'}_x} d\tau \right\|_{L^q_t}$$

Here, we used Minkowski integral inequality, the dispersive bound above and fractional integration in that order. The inequality (5) is proved similarly:
Note that in particular we have the bounds

\[ \| D^{1/4} W_t u_0 \|_{L^4_t L^\infty_x} \lesssim \| u_0 \|_{L^2} . \]

\[ \| W_t u_0 \|_{L^8_t L^8_x} \lesssim \| u_0 \|_{L^2} . \]

**Theorem 3. (Kato Smoothing)**

\[ \| \partial_x W_t u_0 \|_{L^\infty_t L^2_x} \lesssim \| u_0 \|_{L^2} . \]

**Proof.** Writing

\[ \partial_x W_t u_0 = i \int \xi e^{i\xi^2 t + i\xi x} \hat{u}_0(\xi) d\xi \cdot \frac{\eta^{1/3}}{3} \int \eta^{-1/3} e^{i\eta t + i\eta^{1/3} x} \hat{u}_0(\eta^{1/3}) d\eta , \]

we see that (by Plancherel)

\[ \| \partial_x W_t u_0 \|_{L^2_t} = \frac{1}{3} \| \eta^{-1/3} e^{i\eta^{1/3} x} \hat{u}_0(\eta^{1/3}) \|_{L^2_\eta} \| u_0 \|_{L^2_t} = \| u_0 \|_{L^2} . \]

Finally we state without proof Vega’s maximal function inequality:

**Theorem 4.** For any \( s > 3/4 , \)

\[ \| W_t u_0 \|_{L^2_t L^\infty_{x \in [-T,T]} } \lesssim \langle T \rangle^s \| u_0 \|_{H^s} . \]
Remark. Using the inequality $|\xi| \lesssim |\xi - \eta|^s + |\eta|^s$ one obtains the following Leibnitz rule

$$\|D^s(uu_x)\|_{L^2_t L^2_x[0,T]} \lesssim \|u_x D^s u\|_{L^2_t L^2_x} + \|u D^s u_x\|_{L^2_t L^2_x} \lesssim \|u_x\|_{L^2_t L^\infty} \|D^s u\|_{L^2_t L^2_x} + \|u\|_{L^2_t L^\infty} \|D^s u_x\|_{L^2_t L^2_x} \lesssim T^{1/4} \|u_x\|_{L^2_t L^\infty} \|D^s u\|_{L^2_t L^2_x} + \|u\|_{L^2_t L^\infty} \|D^s u_x\|_{L^2_t L^2_x}.$$

In the second and third lines we used Holder’s inequality. Similarly,

$$\|uu_x\|_{L^2_t L^2_x[0,T]} \leq \|u_x\|_{L^4_t L^\infty} \|u\|_{L^2_t L^2_x} \lesssim T^{1/4} \|u_x\|_{L^4_t L^\infty} \|u\|_{L^2_t L^2_x}.$$

Now we are ready to prove that KdV is locally wellposed on $H^s(\mathbb{R})$ for $s > 3/4$. We run a contraction mapping argument in the Banach space $X$ equipped with the norm

$$\|u\|_X = \max \left( \|J^s u\|_{L^\infty_{x,t} L^2_x}, \|u_x\|_{L^2_t L^\infty_x}, \|u\|_{L^2_t L^\infty_x}, \|D^s u_x\|_{L^2_t L^2_x} \right).$$

By Duhamel,

$$u(t) = W_t u_0(x) - \int_0^t W_{t-\tau}(uu_x) d\tau.$$

We start with $\|J^s u\|_{L^\infty_{x,t} L^2_x}$. The estimate for the $L^2$ part is similar. We thus demonstrate the proof for $\|D^s u\|_{L^\infty_{x,t} L^2_x}$. By Duhamel and the remark above

$$\|D^s u\|_{L^\infty_{x,t} L^2_x} \lesssim \|u_0\|_{H^s} + \left\| \int_0^t W_{t-\tau} D^s(uu_x) d\tau \right\|_{L^\infty_{x,t} L^2_x} \lesssim \|u_0\|_{H^s} + \left\| \int_0^t \|D^s(uu_x)\|_{L^2_x} d\tau \right\|_{L^\infty_{t,x}} \lesssim \|u_0\|_{H^s} + T^{1/2} \|D^s(uu_x)\|_{L^2_{x,t}} \lesssim \|u_0\|_{H^s} + T^{3/4} \|u_x\|_{L^4_t L^\infty_x} \|D^s u\|_{L^\infty_{x,t} L^2_x} + T^{1/2} \|u\|_{L^2_t L^\infty} \|D^s u_x\|_{L^2_t L^2_x} \lesssim \|u_0\|_{H^s} + T^{1/2} \|u\|_{X}^2.$$
Similarly (using $\partial_x = DH$, where $H$ is the Hilbert transform)
\[
\|u_x\|_{L_t^4 L_x^\infty} \lesssim \|\partial_x W_t u_0\|_{L_t^4 L_x^\infty} + \left\| \partial_x \int_0^t W_{t-\tau} (u u_x) d\tau \right\|_{L_t^4 L_x^\infty} \\
\leq \| D^{1/4} W_t D^{3/4} H u_0 \|_{L_t^4 L_x^\infty} + \left\| \int_0^t \| D^{1/4} W_t W_{-\tau} D^{3/4} H (u u_x) \|_{L_x^\infty} d\tau \right\|_{L_t^4} \\
\leq \| D^{3/4} H u_0 \|_{L_t^4} + \int_0^T \| D^{1/4} W_t W_{-\tau} D^{3/4} H (u u_x) \|_{L_t^4 L_x^\infty} d\tau \\
\leq \| D^{3/4} H u_0 \|_{L_t^4} + \int_0^T \| D^{3/4} H (u u_x) \|_{L_t^2 L_x^\infty} d\tau \\
\lesssim \| u_0 \|_{H^s} + T^{1/2} \| D^s (u u_x) \|_{L_{x,t}^2} + T^{1/2} \| u u_x \|_{L_{x,t}^2} \\
\lesssim \| u_0 \|_{H^s} + T^{1/2} \| u \|_X^2.
\]

In the forth inequality we used Strichartz and in the last inequality we used the remark above.

Again by Duhamel and the maximal function inequality
\[
\|u\|_{L_t^4 L_x^\infty} \lesssim \| W_t u_0 \|_{L_t^4 L_x^\infty} + \left\| \int_0^t |W_{t-\tau} (u u_x)| d\tau \right\|_{L_t^2 L_x^\infty} \\
\lesssim \| u_0 \|_{H^s} + \int_0^T \| W_t W_{-\tau} (u u_x) \|_{L_x^\infty} d\tau \\
\lesssim \| u_0 \|_{H^s} + \int_0^T \| u u_x \|_{H_x} d\tau \\
\leq \| u_0 \|_{H^s} + T^{1/2} \| D^s (u u_x) \|_{L_{x,t}^2} + T^{1/2} \| u u_x \|_{L_{x,t}^2} \\
\lesssim \| u_0 \|_{H^s} + T^{1/2} \| u \|_X^2.
\]

Finally, we estimate $\| D^s u_x \|_{L_t^\infty L_x^2}$ using Duhamel and Kato smoothing:
\[
\| D^s u_x \|_{L_t^\infty L_x^2} \lesssim \| \partial_x W_t D^s u_0 \|_{L_t^\infty L_x^2} + \left\| \int_0^t |D^s \partial_x W_{t-\tau} (u u_x)| d\tau \right\|_{L_t^\infty L_x^2} \\
\lesssim \| u_0 \|_{H^s} + \int_0^T \| \partial_x W_t W_{-\tau} D^s (u u_x) \|_{L_x^\infty L_t^2} d\tau
\]
\[ \lesssim \|u_0\|_{H^s} + \int_0^T \|uu_x\|_{H^s} d\tau \]
\[ \leq \|u_0\|_{H^s} + T^{1/2} \|D^s((uu_x))\|_{L^2_x,t} + T^{1/2} \|uu_x\|_{L^2_x,t} \]
\[ \lesssim \|u_0\|_{H^s} + T^{1/2} \|u\|_{X}^2. \]

Note that we proved
\[ \|u\|_X \lesssim \|u_0\|_{H^s} + T^{1/2} \|u\|_{X}^2. \]

Therefore one can close the argument in the usual way.

Since we have proved the local well-posedness by a contraction argument it implies continuous dependence on the initial data. Note that \( u \) is continuous at time zero in \( H^s \) since
\[ \|u(t) - u(0)\|_{H^s} \leq \|W_t u_0 - u_0\|_{H^s} + t^{1/2} \|u\|_{X}^2 \rightarrow 0, \text{ as } t \rightarrow 0. \]

This implies that \( u \in C([0, T]; H^s) \) by continuous dependence on initial data as follows:
\[ \|u(t) - u(t + \tau)\|_{H^s} \lesssim \|u(0) - u(\tau)\|_{H^s} \rightarrow 0, \text{ as } \tau \rightarrow 0. \]

This local result and the \( H^1 \) a priori bound coming from the conservation laws extends the solution globally-in-time in \( H^1 \). To see this one uses continuous dependence and obtains a smooth solution that satisfies the conservation law and tends to our solution in the \( H^1 \) norm. Taking limits we show that \( u \) satisfies the \( H^1 \) conservation and the process proceeds as in the case of the Bona-Smith solutions.

One drawback of this method is the fact that it relies on the dispersive estimates that are not true over compact domains, in particular over \( \mathbb{T} \). In the next section following Bourgain we demonstrate a method that implies local well-posedness both on \( \mathbb{R} \) and \( \mathbb{T} \).
3. Restricted norm method of Bourgain

3.1. KdV on $\mathbb{R}$. Let $X^{s,b}$ be the Banach space of functions on $\mathbb{R} \times \mathbb{R}$ (or $T \times \mathbb{R}$) defined by the norm

$$
\|u\|_{X^{s,b}} = \left\| \langle \xi \rangle^s (\tau - \xi^3)^b \hat{u}(\xi, \tau) \right\|_{L^2_{\xi,\tau}} = \left\| W_{-t} u \right\|_{H_x^s H_t^b}.
$$

Since the contraction argument will be in a time interval $[-\delta, \delta]$ with $\delta \leq 1$, we also define the restricted $X^{s,b}$ space, $X^{s,b}_\delta$, as the equivalent classes of functions that agree on $[-\delta, \delta]$ with the norm

$$
\|u\|_{X^{s,b}_\delta} = \inf_{\tilde{u} = u, t \in [-\delta, \delta]} \|\tilde{u}\|_{X^{s,b}}.
$$

The contraction will be for the operator

$$
\Phi u = \eta(t) W_t u_0 - \eta(t) \int_0^t W_{t-s} (uu_x) ds,
$$

where $\eta$ is a $C_0^\infty$ function satisfying $\eta(t) = 1$, $t \in [-1, 1]$. Since $\delta \leq 1$, a fix point of $\Phi$ gives us a solution of KdV on $[-\delta, \delta]$.

We will only discuss the case $s = 0$, which implies that the solution is globally defined for all times due to $L^2$ conservation. Similar ideas can push the local well-posedness to any $s \geq -3/4$.

Lemma 3. For $\delta \leq 1$, $s, b \in \mathbb{R}$, we have

$$
\|\eta(t) W_t u_0\|_{X^{s,b}_\delta} \lesssim \|u_0\|_{H^s}.
$$

Proof.

$$
\|\eta(t) W_t u_0\|_{X^{s,b}_\delta} \leq \|W_t \eta(t) u_0\|_{X^{s,b}} = \|W_{-t} W_t \eta(t) u_0\|_{H_x^s H_t^b} = \|\eta\|_{H^s} \|u_0\|_{H^s} \lesssim \|u_0\|_{H^s}.
$$

The first inequality follows from the definition of the restricted norm and the fact that $W_t$ and $\eta(t)$ commute. \qed
Lemma 4. For any $b > 1/2$, $X^{s,b}$ embeds into $C(\mathbb{R}; H^s)$.

Proof. We will do proof for the real line, the proof is the same on the torus. By Fourier inversion and then a change of variable we have

$$u(t, x) = \int \int \hat{u}(\xi, \tau) e^{it\tau + ix\xi} d\xi d\tau$$

$$= \int \int e^{it\xi^3} \hat{u}(\xi, \tau + \xi^3) e^{it\tau + ix\xi} d\xi d\tau$$

$$= \int e^{it\tau} W_t \psi_t d\tau,$$

where $\hat{\psi}_t(\xi) = \hat{u}(\xi, \tau + \xi^3)$. Therefore for each $t$

$$\|u\|_{H^s} \leq \int \|W_t \psi_t\|_{H^s} d\tau = \int \|\psi_t\|_{H^s} d\tau$$

$$= \int \|\hat{u}(\xi, \tau + \xi^3) \langle \xi \rangle^s \|_{L^2_\xi} d\tau$$

$$\leq \|\langle \tau \rangle^{-b}\|_{L^2_\tau} \|\hat{u}(\xi, \tau + \xi^3) \langle \xi \rangle^s \langle \tau \rangle^b \|_{L^2_\xi L^2_\tau}$$

$$\lesssim \|u\|_{X^{s,b}}.$$

Continuity in $t$ follows from this, continuity of the linear group and the dominated convergence theorem. \qed

Lemma 5. For any $-1/2 < b' < b < 1/2$ and $s \in \mathbb{R}$, we have

$$\|u\|_{X^{s,b'}} \lesssim \delta^{b-b'} \|u\|_{X^{s,b}}.$$

Proof. We will give the proof for $0 \leq b' < b < 1/2$. By duality this implies the inequality for $-1/2 < b' < b \leq 0$ as follows

$$\|u\|_{X^{s,b'}} = \sup_{\|g\|_{X^{-s,-b'}} = 1} \left| \int u g \right| \leq \sup_{\|g\|_{X^{-s,-b'}} = 1} \|u\|_{X^{s,b}} \|g\|_{X^{-s,-b'}} \lesssim \delta^{b-b'} \|u\|_{X^{s,b}}.$$

By combining these two inequalities we get the full range.

To obtain the inequality for $0 \leq b' < b < 1/2$, first note that by replacing $u$ with $J^s u$ we can assume that $s = 0$. Second, by definition of the restricted norm it suffices to prove that (by taking infimum over $\tilde{u}$)

$$\|\eta(t/\delta)\tilde{u}\|_{X^{0,b'}} \lesssim \delta^{b-b'}\|\tilde{u}\|_{X^{0,b}}.$$ 

Suppressing the $\tilde{u}$ notation, we have

$$\|\eta(t/\delta)u\|_{X^{0,b'}} = \|\eta(t/\delta)W_{-1}u\|_{L^2_{s}H^{b'}}.$$ 

Therefore it suffices to prove that

$$\|\eta(t/\delta)f(t)\|_{H^{b'}} \lesssim \delta^{b-b'}\|f\|_{H^b}.$$ 

Using $\langle \tau \rangle^{b'} \leq \langle \tau - \tau_1 \rangle^{b'} + \langle \tau_1 \rangle^{b'}$, we obtain (with $\frac{1}{p_1} = b - b'$, $\frac{1}{p_2} = b$, $\frac{1}{q_1} = \frac{1}{2} + b' - b$, $\frac{1}{q_2} = \frac{1}{2} - b$)

$$\|\eta(t/\delta)f(t)\|_{H^{b'}} \leq \|\eta(t/\delta)J^{b'}f\|_{L^2} + \|fJ^{b'}\eta(t/\delta)\|_{L^2}$$

$$\leq \|\eta(t/\delta)\|_{L^{p_1}} \|J^{b'}f\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|J^{b'}\eta(t/\delta)\|_{L^{q_2}}$$

$$\lesssim \|f\|_{H^b}\|\eta(t/\delta)\|_{L^{p_1}} + \|J^{b'}\eta(t/\delta)\|_{L^{p_2}}$$

$$\lesssim \|f\|_{H^b}\left(\delta^{b-b'} + \|\eta(t/\delta)\|_{H^{\frac{1}{2} - b + b'}}\right)$$

$$\lesssim \delta^{b-b'}\|f\|_{H^b}.$$ 

In the last two inequalities we used Sobolev embedding. □

**Lemma 6.** Let $-\frac{1}{2} < b' \leq 0$ and $b = b' + 1$. Then

$$\left\|\eta(t)\int_0^t W_{t-s} F(s) ds\right\|_{X^{b,b}_{s}} \lesssim \|F\|_{X^{b,b}_{s}}.$$
Proof. As before it suffices to prove the statement with $X^{s,b}$ norms. Note that

$$
\left\| \eta(t) \int_0^t W_{t-s} F(s) ds \right\|_{X^{s,b}} = \left\| \eta(t) \int_0^t W_{-s} F(s) ds \right\|_{H_x^s H_t^b}.
$$

Therefore it suffices to prove that

$$(8) \quad \left\| \eta(t) \int_0^t f(s) ds \right\|_{H^b} \lesssim \| f \|_{H^\nu}.$$  

Writing

$$
\int_0^t f(s) ds = \int \chi_{[0,t]}(s) f(s) ds = \int \chi_{[0,t]}(z) \hat{f}(z) dz = \int \frac{e^{izt} - 1}{iz} \hat{f}(z) dz,
$$

we see that the Fourier transform of the function inside the norm of the left hand side of (8) is

$$
\int \frac{\hat{\eta}(\tau - z) - \hat{\eta}(\tau)}{iz} \hat{f}(z) dz = \int_{|z|<1} + \int_{|z|>1}.
$$

For the contribution of the first integral to the left hand side of (8) we use the mean value theorem to get

$$
\left\| \langle \tau \rangle^b \int_{|z|<1} \right\|_{L^2} \lesssim \left\| \int_{|z|<1} \langle \tau \rangle^b \sup_{|\tau' - \tau|<1} |\hat{\eta}'(\tau')| |\hat{f}(z)| dz \right\|_{L^2}
$$

$$
= \left\| \langle \tau \rangle^b \sup_{|\tau' - \tau|<1} |\hat{\eta}'(\tau')| \right\|_{L^2} \int_{|z|<1} |\hat{f}(z)| dz
$$

$$
\lesssim \sqrt{\int_{|z|<1} |\hat{f}(z)|^2 dz} \lesssim \| f \|_{H^\nu}.
$$

For the contribution of the second integral we use the inequality $\langle \tau \rangle^b \lesssim (\tau - z)^b \langle z \rangle^b$ and Young’s inequality to get

$$
\left\| \langle \tau \rangle^b \int_{|z|>1} \right\|_{L^2} \lesssim \left\| \langle \tau \rangle^b \int \left| \frac{\hat{\eta}(\tau - z) + |\hat{\eta}(\tau)|}{\langle z \rangle} \right| |\hat{f}(z)| dz \right\|_{L^2}
$$

$$
\lesssim \left\| \int \left( \langle \tau - z \rangle^b |\hat{\eta}(\tau - z)| + \langle \tau \rangle^b |\hat{\eta}(\tau)| \right) \langle z \rangle^{1-b} |\hat{f}(z)| dz \right\|_{L^2}.
$$
\[
\lesssim \| \langle \tau \rangle^{b} \hat{\eta} \|_{L^1} \| \langle z \rangle^{b-1} \hat{f} \|_{L^2} + \| \langle \tau \rangle^{b} \hat{\eta} \|_{L^2} \| \langle z \rangle^{-1} \hat{f} \|_{L^1} \\
\lesssim \| \langle z \rangle^{b'} \hat{f} \|_{L^2} + \| \langle z \rangle^{b'} \hat{\eta} \|_{L^2} \| \langle z \rangle^{-1} \hat{f} \|_{L^2} \lesssim \| f \|_{H^{b'}}.
\]

The last inequality follows from the fact that \(-1 - b' < -1/2\). \[\square\]

**Remark.** Note that for \(b = 1/2\), the proof above implies that

\[
\| \eta(t) \int_0^t f(s) ds \|_{H^{1/2}} \lesssim \| f \|_{H^{-1/2}} + \| \langle z \rangle^{-1} \hat{f} \|_{L^1}.
\]

**Lemma 7.** For any \(b > 1/2\) and \(b_1 \geq 1/4\) we have

\[
\| \partial_x u^2 \|_{X^{b_1}} \lesssim \| u \|_{X^{0,b}}^{2}.
\]

**Proof.** Once again we can ignore \(\delta\) dependence and work with \(X^{s,b}\) norms.

By duality, it suffices to prove that

\[
\left| \int g \partial_x u^2 \, dx \, dt \right| \lesssim \| u \|_{X^{0,b}}^{2} \| g \|_{X^{0,b_1}}.
\]

Using the Fourier multiplication formula,

\[
\int f g = \int \hat{f} \hat{g},
\]

and renaming the variables, we write the left hand side as

\[
\left| \int_{\mathbb{R}^2} \xi \hat{u}^2(\xi, \tau) \hat{g}(-\xi, -\tau) d\tau d\xi \right| = \left| \int_{\xi_1 + \xi_2 + \xi_3 = 0} \xi_3 \hat{u}(\xi_1, \tau_1) \hat{u}(\xi_2, \tau_2) \hat{g}(\xi_3, \tau_3) \right|
\]

Using the notation

\[
f_1(\xi, \tau) = f_2(\xi, \tau) = |\hat{u}(\xi, \tau)| \langle \tau - \xi^3 \rangle^b,
\]

\[
f_3(\xi, \tau) = |\hat{g}(-\xi, -\tau)| \langle \tau - \xi^3 \rangle^{b_1},
\]
it suffices to prove that
\[
\int \frac{|\xi|f_1(\xi, \tau) f_2(\xi - \xi_1, \tau - \tau_1) f_3(\xi, \tau)}{\langle \tau - \xi_1 \rangle^b \langle \tau - \tau_1 - (\xi - \xi_1)^3 \rangle^b} d\xi d\tau d\tau_1 \lesssim \prod_{i=1}^3 \|f_i\|_2.
\]

We claim that
\[
\sup_{\xi,\tau} \frac{1}{\langle \tau - \xi^3 \rangle^{2b_1}} \int \frac{1}{\langle \tau_1 - \xi_1^3 \rangle^{2b} \langle \tau - \tau_1 - (\xi - \xi_1)^3 \rangle^{2b}} d\xi_1 d\tau_1 \lesssim 1.
\]

By using the Cauchy-Schwartz inequality in $\xi_1, \tau_1$ integrals and using the claim we estimate (10) by
\[
\int \left( \int f_1^2(\xi_1, \tau_1) f_2^2(\xi - \xi_1, \tau - \tau_1) d\xi_1 d\tau_1 \right)^{1/2} f_3(\xi, \tau) d\xi d\tau \leq \left( \int f_1^2(\xi_1, \tau_1) f_2^2(\xi - \xi_1, \tau - \tau_1) d\xi_1 d\tau_1 \right)^{1/2} \left( \int f_3^2(\xi, \tau) d\xi d\tau \right)^{1/2} = \prod_{i=1}^3 \|f_i\|_2.
\]

It remains to prove (11). Using the estimate (for $b > 1/2$)
\[
\int_\mathbb{R} \frac{1}{\langle x - \alpha \rangle^{2b} \langle x - \beta \rangle^{2b}} dx \lesssim \frac{1}{\langle \alpha - \beta \rangle^{2b}}
\]
in the $\tau_1$ integral we bound (11) by
\[
\sup_{\xi,\tau} \frac{1}{\langle \tau - \xi^3 \rangle^{2b_1}} \int \frac{1}{\langle \tau - \xi_1^3 \rangle^{2b} \langle \tau - \tau_1 - (\xi - \xi_1)^3 \rangle^{2b}} d\xi_1.
\]

Let $x = \tau - \xi_1^3 - (\xi - \xi_1)^3$. Using
\[
\xi_1 = \frac{3\xi^2 \pm \sqrt{3\xi(4\tau - \xi^3 - 4x)}}{6\xi},
\]
we obtain
\[
dx = (3\xi^2 - 6\xi_1) d\xi = \pm \sqrt{3\xi(4\tau - \xi^3 - 4x)} d\xi_1.
\]

Therefore, we can estimate (12) by
\[
\sup_{\xi,\tau} \frac{1}{\langle \tau - \xi^3 \rangle^{2b_1}} \int \frac{1}{\langle x \rangle^{2b} \sqrt{|\xi| \sqrt{|4\tau - \xi^3 - 4x|}}} dx.
\]
Using the inequality (for $b > 1/2$)

$$\int_{\mathbb{R}} \frac{1}{(x)^{2b} \sqrt{|x - \beta|}} dx \lesssim \frac{1}{(\beta)^{1/2}},$$

we obtain (for $b_1 \geq 1/4$)

$$\sup_{\xi, \tau} \frac{|\xi|^{3/2}}{(\tau - \xi^3)^{2b_1} (4\tau - \xi^3)^{1/2}} \lesssim 1.$$

We now run the contraction argument in $X^{a,b}_0$ (with $b > 1/2$ and $\delta$ sufficiently small) for the operator

$$\Phi u = \eta(t) W_t u_0 - \eta(t) \int_0^t W_{t-s}(uu_x) ds.$$

Using the bounds in Lemma 3 and in Lemma 6, we have

$$\|\Phi u\|_{X^{a,b}_0} \lesssim \|u_0\|_{L^2} + \|uu_x\|_{X^{a,b-1}_0}.$$

Now, using Lemma 5 (with $b_1 \geq 1/4$) and then Lemma 7, we obtain

$$\|\Phi u\|_{X^{a,b}_0} \lesssim \|u_0\|_{L^2} + \delta^{1-b-b_1} \|uu_x\|_{X^{a,b-1}_0} \lesssim \|u_0\|_{L^2} + \delta^{1-b-b_1} \|u\|_{X^{1,b}_0}^2.$$  

Therefore we can close the contraction for any $b > 1/2$, $b_1 \geq 1/4$, $1 - b - b_1 > 0$ by choosing $\delta = \delta(\|u_0\|_{L^2}, b, b_1)$ sufficiently small.

3.2. KdV on $\mathbb{T}$. In what follows we will consider mean zero solutions of the KdV equation.

This assumption can be justified as follows (we note that on $\mathbb{R}$ this idea fails): Let $u_t + u_{xxx} + uu_x = 0$ with $u(x, 0) = u_0$. If we integrate the equation in space we obtain $\partial_t \int_\mathbb{T} u(x, t) dx = 0$ and thus

$$\int_\mathbb{T} u(x, t) dx = \int_\mathbb{T} u_0(x) dx.$$
Now set $v(x, t) = u(x - ct, t) - c$ and observe that if $u$ solves KdV with initial data $u(x, 0) = u_0$, then $v$ solves
\[ v_t + 2cv_x + v_{xx} + vv_x = 0 \]
with $v(x, 0) = u_0(x) - c$. Since if we integrate in time we still have that $\partial_t \int_T v(x, t)dx = 0$ we conclude that
\[ \int_T v(x, t)dx = \int_T v_0(x)dx = \int_T u_0(x)dx - 2\pi c. \]
But now we can pick the constant $c$ in such away that $v$ has mean zero,
\[ \int_T v(x, t)dx = 0. \]
Of course $v$ doesn't solve the original KdV anymore but the methods we are developing apply to the new equation step by step. The only difference is that now the multiplier of the linear group is $k^3 - 2ck$ instead $k^3$. Notice that in all calculations that follow this replacement changes nothing.

We start with the Strichartz estimates on the torus.

**Theorem 5.**

\[ i) \|W_tg\|_{L^4_{x,t} \in \mathbb{T}} \lesssim \|g\|_{L^2}, \]
\[ ii) \|W_tg\|_{L^6_{x,t} \in \mathbb{T}} \lesssim \|g\|_{H^*}, \text{ for any } \epsilon > 0. \]

**Proof.** We will only prove ii). The proof of i) is simpler, see the remark below.

First assume that $\hat{g} = 0$ outside $[-N, N]$. We write
\[ \|W_tg\|_{L^6_{x,t} \in \mathbb{T}} = \sum_{k_1,k_2,k_3 \in [-N,N]} \hat{g}(k_1)\hat{g}(k_2)\hat{g}(k_3)\hat{g}(j_1)\hat{g}(j_2)\hat{g}(j_3) \]
\[ \times \int_{\mathbb{T}^2} e^{it(k^3_1+k^3_2+k^3_3-j_1^3-j_2^3-j_3^3)+ix(k_1+k_2+k_3-j_1-j_2-j_3)} dt dx. \]
Performing the integration in $x, t$, we obtain

$$\|W_t g\|_{L^6_{x,t} \cap T} = (2\pi)^2 \sum_{k_1^3 + k_2^3 + k_3^3 = j_1 + j_2 + j_3} \hat{g}(k_1) \hat{g}(k_2) \hat{g}(k_3) \hat{g}(j_1) \hat{g}(j_2) \hat{g}(j_3)$$

$$= (2\pi)^2 \sum_{p,q} \sum_{(k_1, k_2, k_3) \in A_{p,q}} \hat{g}(k_1) \hat{g}(k_2) \hat{g}(k_3) \hat{g}(j_1) \hat{g}(j_2) \hat{g}(j_3)$$

$$= (2\pi)^2 \sum_{p,q} \left| \sum_{(k_1, k_2, k_3) \in A_{p,q}} \hat{g}(k_1) \hat{g}(k_2) \hat{g}(k_3) \right|^2,$$

where

$$A_{p,q} = \{(k_1, k_2, k_3) \in [-N, N]^3 : k_1 + k_2 + k_3 = p, k_1^3 + k_2^3 + k_3^3 = q\}.$$

We claim that for any $\epsilon > 0$, $\#A_{p,q} \lesssim N^\epsilon$. Indeed, writing

$$q - p^3 = k_1^3 + k_2^3 + k_3^3 - (k_1 + k_2 + k_3)^3 = -3(k_1 + k_2)(k_2 + k_3)(k_1 + k_3),$$

we see that quantities $k_i + k_j$ can take at most $N^\epsilon$ different values by using the standard fact that the number of divisors of an integer $N$ is at most $N^\epsilon$ for any $\epsilon > 0$. Since the quantities $k_i + k_j$ uniquely determine $k_1, k_2, k_3$, we are done.

Using the claim and Cauchy-Schwartz, we see that

$$\|W_t g\|_{L^6_{x,t} \cap T} \lesssim \sum_{p,q} \sum_{(k_1, k_2, k_3) \in A_{p,q}} |\hat{g}(k_1)\hat{g}(k_2)\hat{g}(k_3)|^2 \sum_{(k_1', k_2', k_3') \in A_{p,q}} 1$$

$$\lesssim N^\epsilon \sum_{p,q} \sum_{(k_1, k_2, k_3) \in A_{p,q}} |\hat{g}(k_1)\hat{g}(k_2)\hat{g}(k_3)|^2$$

$$= N^\epsilon \sum_{(k_1, k_2, k_3) \in [-N, N]^3} |\hat{g}(k_1)\hat{g}(k_2)\hat{g}(k_3)|^2 = N^\epsilon \|g\|_{L^2}^6.$$

For arbitrary $g \in H^\epsilon$, we write

$$g = \hat{g}(0) + \sum_{n=0}^\infty g_n,$$
where \( \hat{g}_n(j) = \chi_{[2^n, 2^{n+1})}(|j|) \hat{g}(j) \). Taking the \( L^6_{x,t} \) norm of \( W_t g \) and using the inequality above yields the statement.

**Remark.** In the case of \( L^4 \) norm the \( A_{p,q} \) set is defined as

\[
A_{p,q} = \{(k_1, k_2) \in [-N, N]^2 : k_1 + k_2 = p, k_1^3 + k_2^3 = q\}.
\]

Note that this set has cardinality at most 4.

**Corollary 1.** For any space-time function \( u \) and \( b > 1/2 \), we have

\[
i) \qquad \|u\|_{L^4_{x,t} \in \mathbb{T}} \lesssim \|u\|_{X^{0,b}},
\]

\[
ii) \qquad \|u\|_{L^6_{x,t} \in \mathbb{T}} \lesssim \|u\|_{X^{\epsilon,b}}, \text{ for any } \epsilon > 0.
\]

**Proof.** By Fourier inversion and then a change of variable we have

\[
u(t, x) = \int \int \hat{u}(\xi, \tau) e^{it\tau + ix\xi} d\xi d\tau
\]

\[= \int \int e^{it\xi^3} \hat{u}(\xi, \tau + \xi^3) e^{it\tau + ix\xi} d\xi d\tau
\]

\[= \int e^{it\tau} W_t \hat{\psi}_\tau d\tau,
\]

where \( \hat{\psi}_\tau(\xi) = \hat{u}(\xi, \tau + \xi^3) \). Therefore,

\[
\|u\|_{L^4_{x,t} \in \mathbb{T}} \leq \int \|W_t \hat{\psi}_\tau\|_{L^4_{x,t} \in \mathbb{T}} d\tau \lesssim \int \|\hat{\psi}_\tau\|_{L^2_{x,t} \in \mathbb{T}} d\tau
\]

\[= \int \|\hat{u}(\xi, \tau + \xi^3)\|_{L^2_{x,t} \in \mathbb{T}} d\tau
\]

\[\lesssim \|\langle \tau \rangle^{-b} \hat{u}(\xi, \tau + \xi^3) \hat{\langle \tau \rangle}^b\|_{L^2_{x,t} L^2_{\tau} \in \mathbb{T}}
\]

\[
\lesssim \|u\|_{X^{0,b}}.
\]

The proof of ii) is similar. \( \square \)
We now present Bourgain’s refinement of the $L^4$ Strichartz estimate. This refinement allows us to extract a power of $\delta$ in the well-posedness proof.

**Theorem 6.** For any space-time function $u$

$$\|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0,1/3}}.$$  

**Proof.** Let $u = \sum_{m=0}^{\infty} u_{2m}$, where $\hat{u}_{2m} = \hat{u} \chi_{|\tau - k^3| < 2^{m+1}}$. Note that by Plancherel

$$\|u\|_{X^{0,1/3}}^2 \approx \sum_{m=0}^{\infty} 2^{2m/3} \|u_{2m}\|_{L^2_{x,t}}^2. \quad (13)$$

We write

$$\|u\|_{L^4_{x,t}}^2 = \|u^2\|_{L^2_{x,t}} \leq 2 \sum_{m \leq m'} \|u_{2m} u_{2m'}\|_{L^2} = 2 \sum_{m,n \geq 0} \|u_{2m} u_{2m+n}\|_{L^2}.$$

We will estimate

$$\|u_{2m} u_{2m+n}\|_{L^2} = \|\widehat{u_{2m}} \ast \widehat{u_{2m+n}}\|_{L^2_{k,r}} \quad (14)$$

separately in the range $|k| \leq 2^a$ and $|k| > 2^a$.

In the former case, for each $|k| \leq 2^a$, we put the $L^2_r$ norm inside the sum and apply Young’s inequality to obtain

$$\|\widehat{u_{2m}} \ast \widehat{u_{2m+n}}\|_{L^2_{k,r}} \leq \sum_{k_1} \left\| \int \widehat{u_{2m}}(k_1, \tau_1) \widehat{u_{2m+n}}(k - k_1, \tau - \tau_1) d\tau_1 \right\|_{L^2_{k,r}}$$

$$\leq \sum_{k_1} \|\widehat{u_{2m}}(k_1, \cdot)\|_{L^1} \|\widehat{u_{2m+n}}(k - k_1, \cdot)\|_{L^2}$$

$$\lesssim 2^{m/2} \sum_{k_1} \|\widehat{u_{2m}}(k_1, \cdot)\|_{L^2} \|\widehat{u_{2m+n}}(k - k_1, \cdot)\|_{L^2}$$

$$\leq 2^{m/2} \left( \sum_{k_1} \|\widehat{u_{2m}}(k_1, \cdot)\|_{L^2}^2 \right)^{1/2} \left( \sum_{k_1} \|\widehat{u_{2m+n}}(k - k_1, \cdot)\|_{L^2}^2 \right)^{1/2}$$
\begin{align*}
&= 2^{m/2} \| u_{2m} \|_{L^2_{\omega,t}} \| u_{2m+n} \|_{L^2_{\omega,t}}.
\end{align*}

Therefore, taking the $L^2_{|k| \leq 2^a}$ norm we obtain

\begin{equation}
(15)
\| \hat{u}_{2m} * \hat{u}_{2m+n} \|_{L^2_{|k| \leq 2^a, \tau}} \lesssim 2^{\frac{a+m}{2}} \| u_{2m} \|_{L^2_{\omega,t}} \| u_{2m+n} \|_{L^2_{\omega,t}}.
\end{equation}

In the latter case, we have

\begin{align*}
\| \hat{u}_{2m} * \hat{u}_{2m+n} \|_{L^2_{|k| > 2^a, \tau}}
&\leq \left\| \left( \sum_{k_1} \int |\hat{u}_{2m}(k_1, \tau_1)|^2 |\hat{u}_{2m+n}(k - k_1, \tau - \tau_1)|^2 d\tau_1 \right)^{1/2} \left( \chi_m * \chi_{m+n}(k, \tau) \right)^{1/2} \right\|_{L^2_{|k| > 2^a, \tau}},
\end{align*}

where $\chi_m(k, \tau) = \chi_{2^m \leq \tau - k^3 < 2^{m+1}}$. Taking the supremum of the convolution outside the norm we obtain

\begin{align*}
\| \hat{u}_{2m} * \hat{u}_{2m+n} \|_{L^2_{|k| > 2^a, \tau}}
&\leq \| \chi_m * \chi_{m+n} \|_{L^2_{|k| > 2^a, \tau}}^{1/2} \left\| \sum_{k_1} \int |\hat{u}_{2m}(k_1, \tau_1)|^2 |\hat{u}_{2m+n}(k - k_1, \tau - \tau_1)|^2 d\tau_1 \right\|^{1/2}_{L^2_{k, \tau}}
\end{align*}

\begin{align*}
&= \| \chi_m * \chi_{m+n} \|_{L^2_{|k| > 2^a, \tau}}^{1/2} \| u_{2m} \|_{L^2_{\omega,t}} \| u_{2m+n} \|_{L^2_{\omega,t}}.
\end{align*}

To estimate the convolution, we write for fixed $|k| > 2^a$ and $\tau$,

\begin{align*}
\chi_m * \chi_{m+n}(k, \tau) = \sum_{k_1} \int \chi_m(k_1, \tau_1) \chi_{m+n}(k_1, \tau - \tau_1) d\tau_1.
\end{align*}

By the support condition on $\chi_m$ and $\chi_{m+n}$, we have

\begin{align*}
\tau_1 = k_1^3 + O(2^m), \quad \tau - \tau_1 = (k - k_1)^3 + O(2^{m+n}).
\end{align*}

Therefore for each fixed $k_1$, the $\tau_1$ integral is $O(2^m)$. To calculate the number of $k_1$s for which the integral is nonzero, note that

\begin{align*}
\tau = k_1^3 + (k - k_1)^3 + O(2^{m+n}) \implies k^2 - 3k_1 k + 3k_1^2 = \frac{\tau}{k} + O(2^{m+n-a}).
\end{align*}
This implies that
\[3(k_1 - k/2)^2 = \frac{\tau}{k} - \frac{k^2}{4} + O(2^{m+n-a}).\]

Therefore, \(k_1\) takes \(O(2^{m+n-a})\) values. Using this bound, we obtain
\[
\|\hat{u}^{2m} \ast \hat{u}^{2m+n}\|_{L^2_{|k|>2^a,\tau}} \lesssim 2^{\frac{3m+n-a}{4}} \|u^{2m}\|_{L^2_{x,t}} \|u^{2m+n}\|_{L^2_{x,t}}.
\]

Combining (15) and (16), and choosing \(a = \frac{m+n}{3}\), we obtain
\[
\|\hat{u}^{2m} \ast \hat{u}^{2m+n}\|_{L^2_{k,\tau}} \lesssim 2^{\frac{4m+n}{6}} \|u^{2m}\|_{L^2_{x,t}} \|u^{2m+n}\|_{L^2_{x,t}}.
\]

Finally we estimate
\[
\|u\|_{L^4_{x,t}}^2 = \|u^2\|_{L^2_{x,t}} \leq 2 \sum_{m,n \geq 0} \|u^{2m} u^{2m+n}\|_{L^2} \leq 2 \sum_{n \geq 0} 2^{-\frac{n}{6}} \sum_{m \geq 0} 2^{\frac{m}{6}} \|u^{2m}\|_{L^2_{x,t}} 2^{\frac{m+n}{3}} \|u^{2m+n}\|_{L^2_{x,t}}.
\]

By the Cauchy-Schwartz inequality in the \(m\) sum and (13) gives
\[
\|u\|_{L^4_{x,t}}^2 \lesssim \|u\|_{X^{0,1/3}}^2 \sum_{n \geq 0} 2^{-\frac{n}{6}} \lesssim \|u\|_{X^{0,1/3}}^2.
\]

\(\square\)

Unfortunately, for the \(H^s\) local theory of KdV on the torus, the suitable space is \(X^{s,1/2}\) (to kill the derivative one needs to take \(b = 1/2\)). This space does not embedd into \(C(\mathbb{R}; H^s)\).

Therefore we define the space \(Y^s\) via the norm
\[
\|u\|_{Y^s} = \|u\|_{X^{s,1/2}} + \|\langle k \rangle^s \hat{u}(k, \tau)\|_{L^1_{k,\tau}}.
\]

The restricted space \(Y^s_{\delta}\) is defined accordingly.
Note that for each $t$, 
\[
\|u(\cdot, t)\|_{H^s}^2 = \sum_k |(u(\hat{k}), t)|^2 k^{2s} = \sum_k \left| \int_\tau \hat{u}(\hat{k}, \tau) e^{it\tau} d\tau \right|^2 k^{2s} \leq \|u\|_{Y^s}^2.
\]

Continuity in $t$ follows by the dominated convergence theorem. Thus, $Y^s$ embeds into $C(\mathbb{R}; H^s)$.

We will present suitable versions of the lemmas in the previous section for the $Y^s$ norms. As in the previous section, we will ignore the $\delta$ dependence in the proofs.

**Lemma 8.** For any $s$ real,
\[
\|\eta(t)W_tg\|_{Y^s} \lesssim \|g\|_{H^s}.
\]

**Proof.**
\[
\|\eta(t)W_tg\|_{Y^s} = \left\| \eta \hat{W}_t \hat{g}(k, \tau) \right\|_{L^1_t L^1_k} = \left\| \hat{\eta} (\tau + k^3) \hat{g} (k) \right\|_{L^1_t L^1_k} \lesssim \|g\|_{H^s}.
\]

To estimate the Duhamel term we introduce the “dual” space
\[
\|u\|_{Z^s} = \|u\|_{X^{s-1/2}} + \|\hat{u}(k, \tau)\langle k \rangle^{s} \langle \tau - k^3 \rangle^{-1}\|_{L^1_t L^1_k}.
\]

Again, $Z^s_\delta$ is defined accordingly.

**Lemma 9.** We have 
\[
\|\eta(t) \int_0^t W_{t-s} F(s) ds\|_{Y^s_\delta} \lesssim \|F\|_{Z^s_\delta}.
\]

**Proof.** We first estimate the $X^{s,1/2}$ part of the norm:
\[
\|\eta(t) \int_0^t W_{t-s} F(s) ds\|_{X^{s,1/2}} = \|\eta(t) \int_0^t W_{-s} F(s) ds\|_{H^s_x H^s_t}.
\]
\[
= \left\| \eta(t) \int_0^t \left[ W_{-s} F(s) \right] (\hat{k}) ds \langle k \rangle^s \right\|_{\ell^2_k H_t^{1/2}}.
\]

Using (9),
\[
\left\| \eta(t) \int_0^t f(s) ds \right\|_{H_t^{1/2}} \lesssim \| f \|_{H_t^{-1/2}} + \| \langle z \rangle^{-1} \hat{f} \|_{L_t^1},
\]
we estimate this by
\[
\left\| \left[ W_{-s} F(t) \right] (\hat{k}) \langle k \rangle^s \right\|_{\ell^2_k H_t^{1/2}} + \left\| \left[ W_{-s} F(t) \right](k, \tau) \langle k \rangle^s \langle \tau \rangle^{-1} \right\|_{\ell^2_k L_t^1} = \| F \|_{Z_s}.
\]

To estimate the other part of the \(Y^s\) norm, define \(D(x, t) = \eta(t) \int_0^t W_{-s} F(s) ds\). Recall that
\[
\hat{D}(k, \tau) = \int \frac{\hat{\eta}(\tau - z - k^3) - \hat{\eta}(\tau - k^3)}{i z} \hat{F}(z + k^3, k) dz.
\]
Using this we estimate
\[
\| \langle k \rangle^s \hat{D}(k, \tau) \|_{\ell^2_k L_t^1} \leq \left\| \int \frac{\hat{\eta}(\tau - z - k^3) - \hat{\eta}(\tau - k^3)}{i z} \right\|_{L_t^1} \left\| \hat{F}(z + k^3, k) \langle k \rangle^s dz \right\|_{\ell^2_k} \lesssim \left\| \int \langle \tau \rangle^{-1} \hat{F}(z + k^3, k) \langle k \rangle^s dz \right\|_{\ell^2_k} \leq \| F \|_{Z_s}.
\]

We obtained the second line by considering the cases \(|z| < 1\) and \(|z| > 1\) separately. In the former case we used the mean value theorem. \(\square\)

**Theorem 7.** Assume that \(u\) is a space-time function of mean zero for each \(t\), then for \(s > -1/2\) we have
\[
\| \partial_x (u^2) \|_{Z_s} \lesssim \| u \|_{X^{s,1/2}_{-1/2}} \| u \|_{X^{s,1/3}}.
\]

**Proof.** We will give the proof only for the range \(s \in (-1/2, 0]\). The proof is easier for \(s > 0\). We start with the first part of the \(Z^s\) norm:
\[
\| \partial_x (u^2) \|_{X^{s,-1/2}} = \sup_{\| u \|_{X^{s,1/2}_{-1/2}}} \left| \int w \partial_x u^2 dt dx \right|
\]
Using the notation
\[ f_1(k, \tau) = f_2(k, \tau) = |\hat{u}(k, \tau)|\langle k \rangle^s \langle \tau - k^3 \rangle^{1/2}, \]
\[ f_3(k, \tau) = |\hat{w}(-k, -\tau)|\langle k \rangle^{-s} \langle \tau - k^3 \rangle^{1/2}, \]
we estimate the right hand side by
\[ \sum_{k_1+k_2+k_3=0} \int_{\tau_1+\tau_2+\tau_3=0} \frac{\langle k_3 \rangle^{1+s} f_1(k_1, \tau_1) f_2(k_2, \tau_2) f_3(k_3, \tau_3)}{\langle k_1 \rangle^s \langle k_2 \rangle^{s} \langle \tau_1 - k_1^3 \rangle^{1/2} \langle \tau_2 - k_2^3 \rangle^{1/2} \langle \tau_3 - k_3^3 \rangle^{1/2}}. \]
Note that because of the mean zero assumption \( k_j \neq 0 \) in the sum above. We continue by estimating the multiplier, setting \( s = -\rho \in [0, 1/2) \),
\[ \frac{\langle k_1 \rangle^\rho \langle k_2 \rangle^\rho \langle k_3 \rangle^{1-\rho}}{\langle \tau_1 - k_1^3 \rangle^{1/2} \langle \tau_2 - k_2^3 \rangle^{1/2} \langle \tau_3 - k_3^3 \rangle^{1/2}}. \]
Notice that
\[ \tau_1 - k_1^3 + \tau_2 - k_2^3 + \tau_3 - k_3^3 = (k_1 + k_2)^3 - k_1^3 - k_2^3 = 3k_1k_2(k_1 + k_2) = -3k_1k_2k_3. \]
Therefore (using \( k_j \neq 0 \))
\[ \max(\langle \tau_1 - k_1^3 \rangle, \langle \tau_2 - k_2^3 \rangle, \langle \tau_3 - k_3^3 \rangle) \gtrsim \langle k_1 \rangle \langle k_2 \rangle \langle k_3 \rangle. \]
Assume that the largest one is \( \langle \tau_1 - k_1^3 \rangle \), the other cases are similar. The multiplier is estimated by (using \( k_3 = -k_1 - k_2 \))
\[ \frac{\langle k_3 \rangle^{1/2-\rho}}{\langle k_1 \rangle^{1/2-\rho} \langle k_2 \rangle^{1/2-\rho} \langle \tau_2 - k_2^3 \rangle^{1/2} \langle \tau_3 - k_3^3 \rangle^{1/2}} \lesssim \frac{1}{\langle \tau_2 - k_2^3 \rangle^{1/2} \langle \tau_3 - k_3^3 \rangle^{1/2}}. \]
Using this in (17), we obtain
\[ (17) \lesssim \sum_{k_1+k_2+k_3=0} \int_{\tau_1+\tau_2+\tau_3=0} \frac{f_1(k_1, \tau_1) f_2(k_2, \tau_2) f_3(k_3, \tau_3)}{\langle \tau_2 - k_2^3 \rangle^{1/2} \langle \tau_3 - k_3^3 \rangle^{1/2}}. \]
Here we used Fourier multiplication formula and the convolution structure. By using Hölder and then Theorem 6, we estimate this by

\[
\|\mathcal{F}^{-1}(f_1)\|_{L^2_{x,t}} \|\mathcal{F}^{-1}\left(\frac{f_2}{\langle \tau - k^3 \rangle^{1/2}}\right)\|_{L^4_{x,t}} \|\mathcal{F}\left(\frac{f_3}{\langle \tau - k^3 \rangle^{1/2}}\right)\|_{L^4_{x,t}} \\
\leq \|f_1\|_{L^2} \|\mathcal{F}^{-1}\left(\frac{f_2}{\langle \tau - k^3 \rangle^{1/2}}\right)\|_{X^{0,1/3}} \|\mathcal{F}\left(\frac{f_3}{\langle \tau - k^3 \rangle^{1/2}}\right)\|_{X^{0,1/3}} \\
= \|u\|_{X^{s,1/2}} \|u\|_{X^{s,1/3}} \|w\|_{X^{-s,1/3}} \leq \|u\|_{X^{s,1/2}} \|u\|_{X^{s,1/3}} \|w\|_{X^{-s,1/2}}.
\]

We continue with the second part of the $Z^s$ norm. Using duality we write

\[
\left\| \frac{\langle k \rangle^s \hat{u}_x u^2(k, \tau)}{\langle \tau - k^3 \rangle} \right\|_{\ell_k^2 L^1_t} \\
\leq \sup_{\|w\|_{L^{\infty}} = 1} \sum_{k_1 + k_2 + k_3 = 0, \tau_1 + \tau_2 + \tau_3 = 0} \frac{\langle k_3 \rangle^{1+s} \hat{u}(k_1, \tau_1) \hat{u}(k_2, \tau_2) \|w(k_3, \tau_3)\|}{\langle \tau_3 - k_3^3 \rangle} \\
= \sup_{\|w\|_{L^{\infty}} = 1} \sum_{k_1 + k_2 + k_3 = 0, \tau_1 + \tau_2 + \tau_3 = 0} \frac{\langle k_3 \rangle^{1+s} f_1(k_1, \tau_1) f_2(k_2, \tau_2) |w(k_3, \tau_3)|}{\langle k_1 \rangle^s \langle k_2 \rangle^s \langle \tau_1 - k_1^3 \rangle^{1/2} \langle \tau_2 - k_2^3 \rangle^{1/2} \langle \tau_3 - k_3^3 \rangle^{1/2}},
\]

with $f_1$ and $f_2$ as above. By symmetry, we have two cases to consider.

Case 1) $\max(\langle \tau_1 - k_1^3 \rangle, \langle \tau_2 - k_2^3 \rangle, \langle \tau_3 - k_3^3 \rangle) = \langle \tau_1 - k_1^3 \rangle$.

Using (18), the multiplier is bounded by

\[
\frac{\langle k_3 \rangle^{1+s}}{\langle k_1 \rangle^{1+s} \langle k_2 \rangle^{1+s} \langle \tau_1 - k_1^3 \rangle^{1/2} \langle \tau_2 - k_2^3 \rangle^{1/2} \langle \tau_3 - k_3^3 \rangle^{1/2}} \lesssim \frac{1}{\langle \tau_2 - k_2^3 \rangle^{1/2} \langle \tau_3 - k_3^3 \rangle^{1/2}}.
\]

Using this bound as above we estimate norm in this case by

\[
\sup_{\|w\|_{L^{\infty}} = 1} \|f_1\|_{L^2} \|\mathcal{F}^{-1}\left(\frac{f_2}{\langle \tau - k^3 \rangle^{1/2}}\right)\|_{X^{0,1/3}} \|\mathcal{F}\left(\frac{|w|}{\langle \tau - k^3 \rangle}\right)\|_{X^{0,1/3}} \\
= \|u\|_{X^{s,1/2}} \|u\|_{X^{s,1/3}} \sup_{\|w\|_{L^{\infty}} = 1} \left\| \frac{w}{\langle \tau - k^3 \rangle^{2/3}} \right\|_{\ell_k^2 L^1_t}.
\]
\[ \lesssim \|u\|_{X^{s,1/2}} \|u\|_{X^{s,1/3}}. \]

In the last line we used Hölder’s inequality in the \(\tau\) variable.

Case 2) \(\max (\langle \tau_1 - k_3^3 \rangle, \langle \tau_2 - k_3^3 \rangle, \langle \tau_3 - k_3^3 \rangle) = \langle \tau_3 - k_3^3 \rangle\).

Using (18), we estimate
\[ \langle \tau_3 - k_3^3 \rangle \gtrsim \langle k_1 \rangle \langle k_2 \rangle \langle k_3 \rangle \gtrsim \langle k_3 \rangle^2. \]

Therefore we have
\[ \langle \tau_3 - k_3^3 \rangle = \langle \tau_3 - k_3^3 \rangle^{-s} \langle \tau_3 - k_3^3 \rangle^{-s} \langle k_1 \rangle^{-s} \langle k_2 \rangle^{-s} \langle k_3 \rangle^{-s} \left( \langle \tau_3 - k_3^3 \rangle + \langle k_3 \rangle^2 \right)^{-1+s}. \]

Using this, we estimate the multiplier by
\[ \frac{\langle k_3 \rangle^{1+2s}}{\langle \tau_1 - k_3^3 \rangle^{1/2} \langle \tau_2 - k_3^3 \rangle^{1/2} \left( \langle \tau_3 - k_3^3 \rangle + \langle k_3 \rangle^2 \right)^{1+s}}. \]

Using this as above (switching the roles of \(f_1\) and \(w\)), we bound the norm by
\[ \|F^{-1} \left( \frac{f_1}{\langle \tau - k_3^3 \rangle^{1/2}} \right) \|_{X^{s,1/3}} \|F^{-1} \left( \frac{f_2}{\langle \tau - k_3^3 \rangle^{1/2}} \right) \|_{X^{s,1/3}} \sup_{\|w\|_{L_k^2} = 1} \|w(k, \tau) \langle k \rangle^{1+2s} \|_{L^1_{k,r}} \lesssim \|u\|_{X^{s,1/2}} \|u\|_{X^{s,1/3}}. \]

The last line follows from the inequality (using \(s > -1/2\))
\[ \int \frac{1}{(|\tau| + \langle k \rangle^2)^{2+2s}} d\tau \lesssim \frac{1}{\langle k \rangle^{2(1+2s)}}. \]
\[ \square \]
**Corollary 2.** Let $\delta \in (0, 1)$. Assume that $u$ is a space-time function of mean zero for each $t$, then for $s > -1/2$ we have

\[
\|\partial_x(u^2)\|_{Z^{s}_\delta} \lesssim \delta^{\frac{1}{6}} - \|u\|^2_{X^{s,1/2}_\delta}.
\]

We are ready to run the contraction argument. Using Lemma 8, Lemma 9, and Corollary 2, we have

\[
\|\Phi u\|_{Y^s} \lesssim \|u_0\|_{H^s} + \delta^{\frac{1}{6}} - \|u\|^2_{X^{s,1/2}_\delta}.
\]

Thus, one can close the contraction in the space (with $M = M(\|u_0\|_{H^s})$ and $\delta = \delta(M)$)

\[X = \{u : \|u\|_{Y^s} \leq M\}.
\]

**4. Differentiation by parts method on $\mathbb{T}$**

We present below an alternative method for proving local and global well-posedness for $L^2$ data on $\mathbb{T}$. The method can be summarized as changing variables and differentiating by parts in the time variable. This eliminates the derivative in the nonlinearity by replacing it with a higher order pure power nonlinearity. One has to be careful with the resonant terms and do the differentiation by parts twice for this method to work. Moreover to close the contraction we will consider high and low frequencies separately. As in the previous section we will work with mean zero initial data.

Using the Fourier series representation

\[u(x, t) = \sum_{k \in \mathbb{Z}_0} u_k(t)e^{ikx},\]

with

\[u_k := \hat{u}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t, x)e^{-ikx}dx\]
write KdV,
\[ u_t = u_{xxx} + uu_x, \]
on the Fourier side as
\[ \partial_t u_k = \frac{ik}{2} \sum_{k_1 + k_2 = k} u_{k_1} u_{k_2} - i k^3 u_k. \]

Then, using the identity
\[ (k_1 + k_2)^3 - k_1^3 - k_2^3 = 3(k_1 + k_2)k_1 k_2, \]
and the transformation
\[ v_k(t) = u_k(t)e^{i k^3 t} \]
the equation can be written in the form
\[ \partial_t v_k = \frac{ik}{2} \sum_{k_1 + k_2 = k} e^{i 3 k_1 k_2} v_{k_1} v_{k_2}. \]

Integrating both sides in \( t \), we have
\[ v_k(t) - v_k(0) = \int_0^t \frac{ik}{2} \sum_{k_1 + k_2 = k} e^{i 3 k_1 k_2} v_{k_1} v_{k_2} ds. \]

From now on we say \( u \) is a strong solution of KdV on \([-\delta, \delta]\) if \( u \in C([-\delta, \delta]; L^2(\mathbb{T})) \) and if for each \( k \in \mathbb{Z}, \; t \in [-\delta, \delta], \; v_k(t) = u_k(t)e^{ik^3 t} \) satisfies (20).

**Remarks.** i) Solutions as defined above also satisfy (19) for each \( k \) and \( t \). Moreover, we have the following bound
\[ \sup_{t \in [-\delta, \delta]} |\partial_t v_k| \lesssim |k| \]
with the implicit constant depending only on \( \|v\|_{L^\infty_{[-\delta, \delta]}L^2} \).

ii) Below, we will perform the differentiation by parts process. For any given solution, \( v \), the bound on \( \partial_t v_k \) suffices to justify this process. Therefore any given solution is also a solution of the integral equation (25) below. Indeed, it suffices to check that for each \( k \),
one can change the order of sum and differentiation, which follows from the bound above and the mean value theorem.

Since $e^{i\lambda k_1 k_2 t} = \partial_t \left( \frac{1}{3i k_1 k_2} e^{i\lambda k_1 k_2 t} \right)$ differentiation by parts and (19) yields

$$\partial_t v_k = \partial_t \left( \frac{1}{2} ik \sum_{k_1 + k_2 = k} \frac{e^{i\lambda k_1 k_2 t} v_{k_1} v_{k_2}}{3i k_1 k_2} - \frac{1}{2} ik \sum_{k_1 + k_2 = k} \frac{e^{i\lambda k_1 k_2 t}}{3i k_1 k_2} \partial_t (v_{k_1} v_{k_2}) \right)$$

$$= \frac{1}{6} \partial_t \left( \sum_{k_1 + k_2 = k} \frac{e^{i\lambda k_1 k_2 t} v_{k_1} v_{k_2}}{k_1 k_2} \right) - \frac{1}{6} \sum_{k_1 + k_2 = k} \frac{e^{i\lambda k_1 k_2 t}}{k_1 k_2} \left( \partial_t v_{k_1} v_{k_2} + \partial_t v_{k_2} v_{k_1} \right).$$

Note that since $v_0 = 0$, the terms corresponding to $k_1 = 0$ or $k_2 = 0$ are not actually present in the above sums. The last two terms are symmetric with respect to $k_1$ and $k_2$ and thus we can consider only one of them. Using (19) we have

$$\sum_{k_1 + k_2 = k} \frac{e^{i\lambda k_1 k_2 t}}{k_1 k_2} v_{k_1} \partial_t v_{k_2} = \frac{i}{2} \sum_{k = k_1 + k_2} \frac{e^{i\lambda k_1 k_2 t}}{k} v_k - \frac{1}{2} \sum_{k_1 + k_2 = k} \left( \sum_{\mu + \lambda = k_2} e^{i\lambda k_1 k_2 \mu} v_{k_1} v_{k_2} \right)$$

$$= \frac{i}{2} \sum_{k = k_1 + k_2 + \mu + \lambda} \frac{v_k v_{k_1} v_{k_2}}{k} e^{3it(k_1 k_2 \mu + k_1 \mu + k_2 \lambda)}.$$ 

We note that $\mu + \lambda$ can not be zero since $\mu + \lambda = k_2$. Using the identity

$$k k_1 + \mu \lambda = (k_1 + \mu + \lambda) k_1 + \mu \lambda = (k_1 + \mu)(k_1 + \lambda)$$

and thus by renaming the variables $k_2 = \mu$, $k_3 = \lambda$, we have that

$$\sum_{k_1 + k_2 = k} \frac{e^{i\lambda k_1 k_2 t}}{k_1 k_2} v_{k_1} \partial_t v_{k_2} = \frac{i}{2} \sum_{k_1 + k_2 + k_3 = k} \frac{e^{i\lambda k_1 k_2 (k_1 + k_2 + k_3)}}{k_1} v_{k_1} v_{k_2} v_{k_3}.$$ 

All in all we have that

$$\partial_t \left( v_k - \frac{1}{6} B_2(v, v)_k \right) = -\frac{i}{6} R_3(v, v)_k$$

where

$$B_2(u, v)_k = \sum_{k_1 + k_2 = k} \frac{e^{i\lambda k_1 k_2 t} u_{k_1} v_{k_2}}{k_1 k_2}.$$
and 

\[ R_3(u, v, w)_k = \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3 \neq 0}} e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)} \frac{k_1}{k_1} u_{k_1} v_{k_2} w_{k_3}. \]

Now let’s single out the resonant terms for which

\[(k_1 + k_2)(k_3 + k_1) = 0\]

and write

\[ R_3(v, v, v)_k = R_{3r}(v, v, v)_k + R_{3nr}(v, v, v)_k \]

where the subscript \( r \) and \( nr \) stands for the resonant and non-resonant terms respectively. Thus,

\[ R_{3r}(v, v, v)_k = \sum_r \frac{v_{k_1} v_{k_2} v_{k_3}}{k_1} \]

and

\[ R_{3nr}(v, v, v)_k = \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3 \neq 0}} e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)} \frac{k_1}{k_1} u_{k_1} v_{k_2} w_{k_3}, \]

where \( \sum_{\text{nr}} \) means that the sum contains only the terms with non-zero exponents. Similarly, \( \sum^r \) means that the sum contains only the terms with zero exponents. The set for which

\[(21) \quad (k_1 + k_2)(k_3 + k_1) = 0\]

holds is the disjoint union of the following 3 sets

\[ S_1 = \{k_1 + k_2 = 0\} \cap \{k_3 + k_1 = 0\} \Leftrightarrow \{k_1 = -k, \ k_2 = k, \ k_3 = k\}, \]

\[ S_2 = \{k_1 + k_2 = 0\} \cap \{k_3 + k_1 \neq 0\} \Leftrightarrow \{k_1 = j, \ k_2 = -j, \ k_3 = k, \ |j| \neq |k|\}, \]

\[ S_3 = \{k_3 + k_1 = 0\} \cap \{k_1 + k_2 \neq 0\} \Leftrightarrow \{k_1 = j, \ k_2 = k, \ k_3 = -j, \ |j| \neq |k|\}. \]

Thus

\[ R_{3r}(v, v, v)_k = \sum_{\lambda=1}^{3} \sum_{S_\lambda} \frac{v_{k_1} v_{k_2} v_{k_3}}{k_1} = \frac{v_{-k} v_k v_k}{-k} + v_k \sum_{j \in \mathbb{Z}_0} \frac{v_j v_{-j}}{j} + v_k \sum_{j \in \mathbb{Z}_0 \mid |j| \neq |k|} \frac{v_j v_{-j}}{j}. \]
Note that the second and third terms in the sum above are identically zero due to the symmetry relation $j \leftrightarrow -j$. Thus

$$R_{3r}(v, v, v)_k = -\frac{v_k}{k} |v_k|^2,$$

where we used $v_{-k} = \overline{v}_k$. We obtain

$$\partial_t \left( v_k - \frac{1}{6} B_2(v, v)_k \right) = \frac{i}{6k} v_k |v_k|^2 - \frac{i}{6} R_{3nr}(v, v, v)_k.$$

Since the exponent in the last term is not zero we can differentiate by parts one more time and obtain that

$$R_{3nr}(v, v, v)_k = \sum_{k_1+k_2+k_3=k}^{nr} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1} v_{k_2} v_{k_3} =$$

$$\frac{1}{3i} \partial_t B_3(v, v, v)_k - \frac{1}{3i} \sum_{k_1+k_2+k_3=k}^{nr} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} \times$$

$$(\partial_t v_{k_1} v_{k_2} v_{k_3} + \partial_t v_{k_2} v_{k_1} v_{k_3} + \partial_t v_{k_3} v_{k_1} v_{k_2})$$

where

$$B_3(u, v, w)_k = \sum_{k_1+k_2+k_3=k}^{nr} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} u_{k_1} v_{k_2} w_{k_3}.$$

As before we express time derivatives using (19). The terms containing $\partial_t v_{k_2}$ and $\partial_t v_{k_3}$ produce the same expressions and a calculation reveals that

$$\sum_{k_1+k_2+k_3=k}^{nr} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} \times$$

$$(\partial_t v_{k_1} v_{k_2} v_{k_3} + \partial_t v_{k_2} v_{k_1} v_{k_3} + \partial_t v_{k_3} v_{k_1} v_{k_2}) = iB_4(v, v, v, v)_k$$

where

$$B_4(u, v, w, z)_k = \frac{1}{2} B_4^1(u, v, w, z)_k + B_4^2(u, v, w, z)_k.$$
From now on $\sum^*$ means that the sum is over all indices for which the denominator do not vanish. The term corresponding to \( \partial_t v_k \) is
\[
B^1_4(u, v, w, z)_k = \sum^* \frac{e^{it\psi(k_1, k_2, k_3, k_4)}}{(k_1 + k_2)(k_1 + k_3 + k_4)(k_2 + k_3 + k_4)} u_{k_1} v_{k_2} w_{k_3} z_{k_4},
\]
and the sum of the terms corresponding to \( \partial_t v_{k_2} \) and \( \partial_t v_{k_3} \) is
\[
B^2_4(u, v, w, z)_k = \sum^* \frac{e^{it\psi(k_1, k_2, k_3, k_4)}}{k_1(k_1 + k_2)(k_1 + k_3 + k_4)(k_2 + k_3 + k_4)} u_{k_1} v_{k_2} w_{k_3} z_{k_4}.
\]
The phase function \( \psi \) will be irrelevant for our calculations since it is going to be estimated out by taking absolute values inside the sums. Hence for \( R_{3nr}(v, v, v) \) we have:
\[
R_{3nr}(v, v, v)_k = \frac{1}{3i} \partial_t B_3(v, v, v)_k - \frac{1}{3} \left( \frac{1}{2} B^1_4(v, v, v, v)_k + B^2_4(v, v, v, v)_k \right).
\]
If we put everything together and combining the two \( B_4 \) terms in one we obtain
\[
(22) \quad \partial_t (v_k - B(v)) = \frac{iv_k|v_k|^2}{6k} + \frac{i}{18} B_4(v)_k,
\]
where
\[
B(v)_k = -\frac{1}{6} \sum_{k_1+k_2=k} \frac{e^{ik_1k_2t} v_{k_1} v_{k_2}}{k_1 k_2} + \frac{1}{18} \sum_{k_1+k_2+k_3=k} \frac{e^{ik_1+k_2+k_3} v_{k_1} v_{k_2} v_{k_3}}{k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)}
\]
and
\[
B_4(v)_k = \frac{1}{2} \sum_{k_1+k_2+k_3+k_4=k} \frac{e^{i\psi(k_1,k_2,k_3,k_4)} (2k_3 + 2k_4 + k_1) v_{k_1} v_{k_2} v_{k_3} v_{k_4}}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)}.
\]
Integrating (22) from 0 to \( t \), we obtain
\[
(23) \quad v_k(t) = v_k(0) + B(v)(t) - B(v)(0) + \int_0^t \left( \frac{iv_k|v_k|^2}{6k} + \frac{i}{18} B_4(v)_k \right)(s) ds.
\]
Note that if we integrate the original equation (19), we obtain
\[
(24) \quad v_k(t) = v_k(0) + \int_0^t \frac{ik}{2} \sum_{k_1+k_2=k} e^{ik_1k_2} v_{k_1} v_{k_2} ds.
\]
Fix $N$ large to be determined later. Define the operator $T$ as follows

$$T(v)(t) = \begin{cases} 
  v_k(0) + B(v)(t) - B(v)(0) + \int_0^t \left( \frac{iv_k|v|^2}{6k} + \frac{i}{18} B_4(v) \right)(s) ds & |k| > N \\
  v_k(0) + \int_0^t \frac{ik}{2} \sum_{k_1+k_2=k} e^{i3kk_1k_2s} v_{k_1}v_{k_2} ds & |k| \leq N
\end{cases}$$

(25)

Proposition 1.

$$\|B(v)\|_{\ell^2_{|k|>N}} \lesssim \frac{1}{N^{1/4}} \left( \|v\|_{L^2}^2 + \|v\|_{L^3}^3 \right)$$

(26)

$$\|B_4(v)\|_{\ell^2_{|k|>N}} \lesssim \|v\|_{L^2}^4$$

(27)

$$\left\| \frac{v_k}{k} v_k^2 \right\|_{\ell^2_{|k|>N}} \lesssim \frac{1}{N} \|v\|_{L^2}^3$$

(28)

$$\left\| k \sum_{k_1+k_2=k} e^{i3kk_1k_2s} v_{k_1}v_{k_2} \right\|_{\ell^2_{|k|\leq N}} \lesssim N^{3/2} \|v\|_{L^2}^2$$

(29)

Using this proposition, we now prove that $T$ is a contraction on

$$X = \{ v \in C([-\delta, \delta]; \ell^2) : \|v\|_{L^\infty_{[-\delta, \delta]} \ell^2} \leq M \},$$

where $N, M, \delta$ depends on $\|u_0\|_{L^2}$. Indeed,

$$\|Tv\|_{L^\infty_{[-\delta, \delta]} \ell^2} \lesssim \|v(0)\|_{L^2} + \frac{1}{N^{1/4}} \left( \|v\|_{L^\infty_{[-\delta, \delta]} \ell^2}^2 + \|v\|_{L^\infty_{[-\delta, \delta]} \ell^2}^3 \right)$$

$$+ \delta \frac{1}{N} \|v\|_{L^\infty_{[-\delta, \delta]} \ell^2}^3 + \delta \|v\|_{L^\infty_{[-\delta, \delta]} \ell^2}^4 + \delta N^{3/2} \|v\|_{L^\infty_{[-\delta, \delta]} \ell^2}^2$$

First choosing $M$ large depending on $\|u_0\|_{L^2}$, then $N$ large depending on $M$, and finally $\delta$ small depending on $N$ and $M$, we see that $T$ is a contraction on $X$. This gives us a unique solution in $X$ and continuous dependence on initial data for the equation $Tv = v$.

Note that the smooth solutions of KdV, which exists by the Bona-Smith method presented above, also solves this equation by the remark in the beginning of this section.

Given $L^2$ initial data, $v(0)$, we need to prove that the solution, the fixed point $v$ of $T$, also solves KdV. To do this, we approximate $v(0)$ by a smooth sequence, $v_n(0)$, and obtain
the corresponding solutions $v_n$ of KdV. Since $v_n$ also solve the new equation, by continuous
dependence on initial data $v_n$ converges to $v$ in $C([-\delta, \delta]; \ell^2)$. Therefore, for each fixed $k$,
taking the limit as $n \to \infty$ in (24), we see that $v$ also satisfies (24), and is a solution of
KdV.

We now prove the uniqueness of the solution of KdV for a given initial data in $L^2$. Let
$v_1, v_2 \in C([-\delta, \delta]; L^2(\mathbb{T}))$ be two solutions of KdV with the same initial data. By the
remark in the beginning of this section, $v_1$ and $v_2$ are fixed points of the equation $Tv = v$. Therefore, $v_1 = v_2$.

**Remark.** Uniqueness as it is proved above is known as “unconditional uniqueness” in
the literature. Note that the methods used in the previous two sections give uniqueness only
in a proper subset of $C([-\delta, \delta], H^s)$.

We now prove Proposition 1.

**Proof of Proposition 1.** We start with (26). For $|k| > N$, we have

$$|B(v)_k| \lesssim \frac{1}{N} \sum_{k_1 + k_2 = k} \frac{|v_{k_1}| |v_{k_2}|}{|k_2|} + \frac{1}{N^{1/4}} \sum^*_{k_1 + k_2 + k_3 = k} \frac{|v_{k_1}| |v_{k_2}| |v_{k_3}|}{|k_1| |k_2|^{3/4}}.$$  

The first one follows assuming by symmetry that $|k_1| \gtrsim |k|$, while the second follows using

$$|(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)| \gtrsim \max(|k_1|, |k_2|, |k_3|) \gtrsim |k_2|^{3/4} |k_1|^{1/4}.$$  

Taking the $\ell^2$ norm, we have

$$\|B(v)\|_{\ell^2_{|k| > N}} \lesssim \frac{1}{N} \|v_k\| \|v_k\|_{\ell^2} + \frac{1}{N^{1/4}} \|v_k\| \|v_k\|_{\ell^{3/4}} \|v_k\|_{\ell^2}$$  

$$\lesssim \frac{1}{N} \|v\|_{\ell^2} \|v_k\|_{\ell^1} + \frac{1}{N^{1/4}} \|v\|_{\ell^2} \|v_k\|_{\ell^1} \|v_k\|_{\ell^{3/4}}$$  

$$\lesssim \frac{1}{N^{1/4}} (\|v\|_{\ell^2}^2 + \|v\|_{\ell^2}^3),$$  

where we used Young’s inequality and Cauchy Schwartz.
The inequality (28) is immediate since $\ell^2 \subset \ell^\infty$.

To prove (29), we note

$$\|k \sum_{k_1+k_2=k} e^{i3kk_1k_2}s v_{k_1} v_{k_2}\|_{\ell^2_{|k| \leq N}} \lesssim \|v*v\|_{\ell^\infty} \|k\|_{\ell^2_{|k| \leq N}} \lesssim N^{3/2} \|v\|_{\ell^2}^2$$

It remains to prove (27). We estimate $B_4$ as

$$|B_4(v)_k| \lesssim \sum_{k_1+k_2+k_3+k_4=k} \left| \frac{|v_{k_1}v_{k_2}v_{k_3}v_{k_4}|}{|k_1+k_2||k_1+k_3+k_4||k_2+k_3+k_4|} \right| + \sum_{k_1+k_2+k_3+k_4=k} \left| \frac{|v_{k_1}v_{k_2}v_{k_3}v_{k_4}|}{|k_1||k_1+k_2||k_2+k_3+k_4|} \right|.$$  

We will estimate the first line, the same method works for the second one. By duality it suffices to estimate

$$\sup_{\|h\|_{\ell^2}} \sum_{k_1,k_2,k_3,k_4}^* \left| \frac{|v_{k_1}v_{k_2}v_{k_3}v_{k_4}|}{|k_1+k_2||k_1+k_3+k_4||k_2+k_3+k_4|} \right| \lesssim \|v\|_{\ell^2}^4.$$  

The estimate for the first sum follows by summing in the order $k_2, k_3, k_1, k_4$. For the second we sum in the order $k_1, k_4, k_2, k_3$. 

$\square$