Let \( f_n(x) = \begin{cases} \gamma_n & \text{for } x = \frac{1}{2} \\ \gamma_n & \text{for } \frac{1}{2} - \frac{1}{n} \leq x < \frac{1}{2} \\ -\gamma_n & \text{for } \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} \)

where \( \lim_{n \to \infty} \gamma_n = \infty \).

a) As \( n \to \infty \), \( \frac{1}{2} - \frac{1}{n} \to \frac{1}{2} \) and thus

\[ |f_n(x)| - 0| = |f_n(x)| \to 0 \quad \text{as } n \to \infty \text{ since } \ f_n\left(\frac{1}{2}\right) = 0. \]

b) The convergence is not uniform since

\[ \max_{x \in \mathbb{R}} |f_n(x)| - 0| = \max_{x \in \mathbb{R}} |f_n(x)| = |\gamma_n| \quad \text{But } \lim_{n \to \infty} \gamma_n = \infty \]

and thus \( \left( |\gamma_n| \leq |\gamma_n| \right) \lim_{n \to \infty} |\gamma_n| = \infty. \)

Thus although \( f_n(x) \to 0 \) pointwise \( f_n(x) \not\to 0 \) uniformly.

c) \[ \left\| f_n(x) - 0 \right\|^2 = \int_{-\infty}^{\infty} |f_n(x)|^2 \, dx = \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2}} |f_n(x)|^2 \, dx + \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{n}} |f_n(x)|^2 \, dx \]

\[ = |\gamma_n|^2 \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2}} \, dx + |\gamma_n|^2 \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{n}} \, dx = \frac{|\gamma_n|^2}{n} + \frac{|\gamma_n|^2}{n} = \frac{2|\gamma_n|^2}{n} \]

If \( \gamma_n = n^{1/3} \Rightarrow |\gamma_n|^2 = n^{2/3} \) and the
\[ \| f_{n}(x) - 0 \|_2^2 = \frac{2n^{2/3}}{\eta} = \frac{2}{n^{1/3}} \Rightarrow \| f_{n}(x) - 0 \|_2^2 = \frac{\sqrt{2}}{\eta^{1/6}} \rightarrow 0 \]

\[ \text{as } n \rightarrow \infty. \]

But if \( \gamma_n = 0 \) then \[ \| f_{n}(x) - 0 \|_2^2 = \frac{2n^2}{\eta} = 2n \]

and \( \lim_{n \rightarrow \infty} \| f_{n}(x) \|_2 = \infty. \)

\#9 \( f(x) \) defined on \((-\ell, \ell)\) satisfies \( f(-\ell) = f(\ell) \)

We know that \( a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) dx \)

\( b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right) dx \)

\( a'_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f'(x) \cos \left( \frac{n\pi x}{\ell} \right) dx = \frac{1}{\ell} \left[ f(x) \cos \left( \frac{n\pi x}{\ell} \right) \right]_{-\ell}^{\ell} \)

\[ - \frac{1}{\ell} \int_{-\ell}^{\ell} \left[ \frac{n\pi}{\ell} \sin \left( \frac{n\pi x}{\ell} \right) \right] dx \]

\[ = \frac{1}{\ell} \left[ f(\ell) \cos \left( \frac{n\pi \ell}{\ell} \right) - f(-\ell) \cos \left( \frac{n\pi (-\ell)}{\ell} \right) \right] + \frac{n\pi}{\ell} \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right) dx \]

\[ = \frac{n\pi}{\ell} b_n \]
\[ b_n' = \frac{i}{\ell} \int_{-\ell}^{\ell} f'(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx = \frac{i}{\ell} \left[ \frac{f(x)}{n\pi} \right]_{-\ell}^{\ell} \]

\[ -\frac{1}{\ell} \int_{-\ell}^{\ell} \frac{n\pi}{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx = -\frac{n\pi a_n}{\ell} \]

\# 10 \quad a_n' = \frac{n\pi}{\ell} b_n \quad \Rightarrow \quad b_n = \frac{\ell}{n\pi} a_n'\]

\[ b_n' = -\frac{n\pi}{\ell} a_n \quad \Rightarrow \quad a_n = -\frac{\ell}{n\pi} b_n' \]

\[ |a_n| + |b_n| = \frac{\ell}{\pi n} \left( |a_n'| + |b_n'| \right) = \frac{\ell}{n\pi} \left( \frac{1}{\ell} \int_{-\ell}^{\ell} f'(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx \right) \]

Since \( f \) is \( C^1([-\ell, \ell]) \), we have that there exists \( M \) such that \( |f'(x)| \leq M \).

Since \( |\cos(x)|, |\sin(x)| \leq 1 \), we have

\[ |a_n| + |b_n| \leq \frac{1}{n\pi} \left[ \int_{-\ell}^{\ell} |f(x)| \cos \left( \frac{n\pi x}{\ell} \right) \, dx + \int_{-\ell}^{\ell} |f'(x)| \sin \left( \frac{n\pi x}{\ell} \right) \, dx \right] \]

\[ \leq \frac{M2\ell}{n\pi} + \frac{2\ell}{\pi} \leq \frac{4\ell M}{\pi} \frac{1}{\eta} = \frac{k}{\eta} \quad \text{for} \quad k = \frac{4\ell M}{\pi} \]
The sine series of \( f(x) = x \) on \((0, \ell)\) is the example 3 on page 109. We have seen that \( a_n = (-1)^{n+1} \frac{2\ell}{\ell \pi} \)

and thus \( X = \sum_{n=1}^{\infty} \frac{\ell}{\nu \pi} (-1)^{n+1} \frac{2\ell}{\ell \pi} \sin \left( \frac{n \pi x}{\ell} \right) \)

Parseval's identity reads \( \sum_{n=1}^{\infty} |a_n|^2 \int_0^\ell |X_n(x)|^2 \, dx = \int_0^\ell |f(x)|^2 \, dx \)

But \( |a_n|^2 = \left( \frac{2\ell}{\nu \pi} \right)^2 \), \( \int_0^\ell |\sin \left( \frac{n \pi x}{\ell} \right)|^2 \, dx = \frac{\ell}{2} \)

\( \int_0^\ell x \, dx = \frac{\ell^3}{3} \). Thus

\( \sum_{n=1}^{\infty} \frac{4 \ell^2}{n^2 \pi^2} \cdot \frac{\ell}{2} = \frac{\ell^3}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \)