1) Complete the proof of elliptic regularity we proved in class. In particular, if $L$ is a constant coefficient elliptic operator of order $m$, $u \in D'$ and $Au \in C_c^\infty(\Omega)$, then $u \in C^\infty(\Omega)$ where $\Omega$ is open in $\mathbb{R}^n$.

2) Define the restriction map $R : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-k})$ by

$$Rf(y) = f(y, 0)$$

Show that if $s > \frac{k}{2}$, then $R$ extends to a bounded map from $H^s(\mathbb{R}^n)$ to $H^{s - \frac{k}{2}}(\mathbb{R}^{n-k})$. In other words, $\|Rf\|_{H^{s - \frac{k}{2}}} \leq C \|f\|_{H^s}$.

3) Let $s > \frac{n}{2}$. Prove that $H^s$ is an algebra and more precisely that

$$\|fg\|_{H^s} \leq C \|f\|_{H^s} \|g\|_{H^s}$$

by following and proving

i) $(1 + |x|^2)^{\frac{1}{2}} \hat{f g}(\xi) = \int K(\xi, \eta) u_x(\xi - \eta) \eta \, d\eta$ where

$$u_x, \eta \in L^2$$

and

$$K(\xi, \eta) = \frac{1}{(1 + |\xi|^2)^{\frac{1}{2}} (1 + |\xi - \eta|^2)^{-\frac{1}{2}} (1 + |\eta|^2)^{-\frac{1}{2}}}$$

ii) $K(\xi, \eta) \leq \begin{cases} C \frac{1}{(1 + |\eta|^2)^{\frac{1}{2}}} & \text{if } |\xi - \eta| > \frac{1}{2} |\xi| \\ C \frac{1}{(1 + |\xi - \eta|^2)^{\frac{1}{2}}} & \text{if } |\xi - \eta| < \frac{1}{2} |\xi| \end{cases}$
1.2. Show that if \( u, v \in L^2 \) and \( w(x) = \int (1 + |m|^2)^{-\frac{3}{2}} u(x-n) v(n) \, dn \) then
\[
\|w\|_{L^2} \leq C \|u\|_{L^2} \|v\|_{L^2}
\]

4. Let \( K \subset \mathbb{R}^n \) compact. (Follow my notes in class) Let \( V_1, \ldots, V_N \) bounded and open sets such that \( K \subset V_j \). Then there exists \( \varphi_1, \ldots, \varphi_N \) with \( \varphi_j \in C_0^\infty(V_j) \) such that \( \sum \varphi_j = 1 \) on \( K \).

5. In the following \( U \subset \mathbb{R}^n \) open, bounded with smooth boundary and all operators satisfy the uniform ellipticity condition.

Let \( \Delta u = \sum_{i=1}^{n} (a_{ij} u_{x_i}) x_j + c \). Prove that there exists \( \mu > 0 \) such that
\[
\|u\|_{L^4} \leq \mu \|u\|_{L^2}.
\]

7. \( B[.,.] \) satisfies the hypothesis of Lax-Milgram theorem, provided that
\[
c(x) \geq -\mu.
\]

6. Prove that, given \( f \in L^2(U) \), there exists a unique weak solution of
\[
\Delta u = f \text{ in } U
\]
\[
u u = 0 \text{ on } \partial U
\]

Hint: Consider \( H_0^2(U) \).
7a) If \( U \) is connected then the only smooth solutions of \( \Delta u = 0 \) in \( U \)

\[ \frac{\partial u}{\partial n} = 0 \text{ on } \partial U \]

are \( u = \text{constant} \).

b) Prove that \( f u(x) \) is a weak solution of \( -\Delta u = f \) in \( U \)

\[ \frac{\partial u}{\partial n} = 0 \text{ on } \partial U \]

then \( \int_U f \, dx = 0 \).