HW2 Solutions

(1) a) \(\Rightarrow\) b) \(d)\): \(\dim M\) is relatively compact and b) not true. Choose \(u \in M\).

Since there is \(E_0 > 0\) such that \(M\) has no finite \(E_0\)-net, there exists \(u_2 \in M\) such that \(\|u_2 - u_1\| > E_0\). Furthermore, \(\exists u_3 \in M\) such that \(\|u_3 - u_2\| > E_0\) and \(\|u_3 - u_1\| > E_0\). Continuing, we get \(\\{u_n\} \in M\) such that \(\|u_n - u_m\| > E_0\) for all \(n \neq m\).

But then \(\{u_n\}\) cannot contain any convergent subsequence.

b) \(\Rightarrow\) a) let \(\{u_n\} \in M\) and \(\lim E = E_1\). Then \(\\{V_j\} \in M\) such that \(\|u_n - V_j\| \leq 1\)

for infinitely many \(n\). Then for a subsequence \(u^{(1)}_{n_j}\) we have

\[\|u^{(1)}_{n_j} - V_j\| \leq 1\] for all \(n\). In addition, \(\|u^{(1)}_{n_j} - u^{(1)}_m\| \leq \|u^{(1)}_{n_j} - V_j\| + \|V_j - u^{(1)}_m\| \leq 2\).

Continue for \(E = \frac{1}{n}\) we obtain the sequences \(u^{(1)}_1, u^{(1)}_2, u^{(1)}_3\), \(E = 1\)

\(u^{(2)}_1, u^{(2)}_2, u^{(2)}_3\), \(E = \frac{1}{2}\)

with the following properties: For each \(K = 1, 2, \ldots\), \(u^{(K+1)}_{n_j}\) is a subsequence of \(u^{(K)}_{n_j}\) and \(\|u^{(K)}_{n_j} - u^{(K)}_m\| \leq \frac{2}{K}\) for all \(n, m\).

Consider the diagonal sequence \(V_n = u^{(n)}_{n_j}\). Then \(\|V_n - V_m\| \leq \frac{2}{n}\) and it is Cauchy. Since \(X\) is Banach \(\{V_n\}\) is convergent.
(2) The first part I did in class. To prove that is closed let 
\[ x \in D(T) \text{ and } \lim_{n \to \infty} x_n = x \]
\[ T x_n = x_n' \to y \]

Convergence in \( C[0,1] \) is uniform and
\[ \int_0^t y(s) \, ds = \lim_{n \to \infty} \int_0^t x_n'(s) \, ds = \lim_{n \to \infty} x_n'(t) - x_n'(0) = x(t) - x(0) \]

Thus \( x(t) = x(0) + \int_0^t y(s) \, ds \). Then \( x \in D(T) \) and \( x' = y \).

(3) \( i) \Rightarrow ii) \) If \( T \) is invertible then \( \text{Im} T = X \). In addition
\[ \| x \| = \| T^{-1} (T x) \| \leq \| T^{-1} \| \| T x \| \]
We know \( T' \) is bounded by the open mapping theorem and thus
\[ \| T x \| \geq \| T' \|^{-1} \| x \| \]

\( ii) \Rightarrow i) \) Since \( \text{Im} T \) is dense in \( X \) we have \( \text{Im} T = X \). But \( \text{Im} T \) is closed and thus \( \text{Im} T = X \). Now let \( x \in X \) then
\[ \| x \| \leq \frac{1}{\alpha} \| T x \| = 0 \Rightarrow x = 0 \]

Thus \( T \) is bounded, 1-1 and onto. By the open mapping theorem \( T \) is invertible.

Notice that we know the existence of unique \( S : x \Rightarrow x \) such that \( S o T = I \) and \( T o S = I \).

The open mapping theorem shows that \( S \) is bounded and the \( T \) invertible.
(4) Trivial by definition

(5) Assume $T$ is not bounded. Then there exist $\{x_n\}$ with $\|x_n\| = 1$ such that

$\|Tx_n\| \geq n$. But $\{x_n\}$ is bounded and $T$ compact, thus there exists $\{Tx_n\}$

which converges which contradicts $\|Tx_n\| \geq n$.

(6) Let $\{x_n\}$ is a bounded sequence in $X$. If $S$ is compact then $\exists \{x_{n_j}\}$

such that $Sx_{n_j}$ converges. But then $TSx_{n_j}$ converges by continuity of $T$.

Thus $TS$ is compact.

Now let $T$ compact. Let $\{x_n\}$ bounded $\Rightarrow \{Sx_n\}$ is bounded since $S$ is

bounded. Then $\exists Sx_{n_j}$

$TSx_{n_j}$ is convergent and thus $TS$ is

compact.

(7) a) The set $K = \{x \in X : \|x\| = 1\}$ is not compact if $X$ is infinite dimensional

(The easy proof can be found in any book of Functional Analysis)

Thus there exists a sequence of unit vectors $\{x_n\}$ in $X$ which does not have any

convergent subsequence. Since $Ix_n = x_n$, $I$ is not compact

b) If $T$ is invertible then $T^{-1}$ is bounded. $T$ is compact and $T^{-1}$ is by HW #6

$T^{-1}T = I$ is compact, contradiction
If $T$ has finite rank then $\text{Im} \ T$ is finite dimensional. If $\{x_n\}$ is bounded then by the boundedness of $T$, $\{Tx_n\}$ is bounded in a finite dimensional space.

By the Bolzano-Weierstrass theorem, this sequence contains a convergent subsequence and thus $T$ is compact.

If $X$ is finite then $\text{Im} \ T \subset \text{Im} \ X$ is finite and by part (i) $T$ is compact.

If $\dim X$ is finite then $\dim(T) \leq \dim X$ thus $\dim(T)$ is finite. By part (i) we are done.

Since $T$ is linear, any linearly independent set in $T(X)$ has inverse images that are linearly independent on $X$.

Let $B(0,N)$ be ball around zero of radius $N$. Then $X = \bigcup_{n \in \mathbb{N}} B(0,N)$. Since $B(0,N)$ is bounded for each fixed $N$, $T(B(0,N))$ is relatively compact subset of $\text{Im} \ T$. By #1 it has a finite $\varepsilon$-net. Thus for any $n \in \mathbb{N}$, $n \in \mathbb{N}$ there exists $v_j^n, j = 1, \ldots, K$ such that

$$ T(B(0,N)) \subset \bigcup_{n \in \mathbb{N}} B(v_j, \frac{1}{n}) $$. Define $A = \bigcup_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}, j = 1, \ldots, K} v_j^n$.

$A$ is countable and it is not hard to see that $A$ is dense in $\text{Im} \ T$.

The $\text{Im} \ T$ is separable.