Some answers to HW 1.

1b) $\ell^p$ is a separable metric space for $1 \leq P < \infty$

We note that $\ell^\infty$, which is the space of all bounded sequences of complex numbers with the metric

$$d(x, y) = \sup_{j \in \mathbb{N}} |x_j - y_j|$$

is not separable.

To see this let $y = (y_1, y_2, \ldots, y_n, \ldots)$ be a sequence of zeros and ones, $y \in \ell^\infty$ and they are uncountably many such sequences. To see this associate $y$ with

$\tilde{y}$ a real number in $[0, 1]$ whose binary representation is

$$\frac{y_1}{2^1} + \frac{y_2}{2^2} + \ldots$$

Since $[0, 1]$ is uncountable and any number inside $[0, 1]$ has a binary representation, the set of sequences of zeros and ones is uncountable.

If we let each of these sequences be the center of a ball with radius less than $1/2$, we have uncountably many non-intersecting balls. Now take any $M$ dense in $\ell^\infty$. Each ball contains an element of $M$ and thus $\ell^\infty$ is uncountable.

For $1 \leq P < \infty$ consider the countable set $\mathcal{Q}$ of sequences of the form

(Without loss of generality consider real-valued sequences)

$$y = (y_1, y_2, \ldots, y_n, 0, \ldots, 0, \ldots)$$

and $y_i$ rational.

Let $x = (x_i) \in \ell^P$ be such for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^\infty |x_j|^P < \frac{\varepsilon}{2}$$

Then for each $x_j$ there is $y_j \in \mathcal{Q}$ such that

$$|x_j - y_j| < \frac{\varepsilon}{2}$$

$$\mathcal{Q} = \mathbb{R}$$

and for each $x_j$ there is $y_j \in \mathcal{Q}$ such that

$$\left(\frac{\varepsilon}{2N}\right)^{1/P}$$

This we can find $y \in \mathcal{M}$

$$\sum_{j=1}^N |x_j - y_j|^P < \frac{\varepsilon}{2}$$
and there \( d(x,y)^p \leq \epsilon^p \). Thus for any \( x \in \ell^p \) we can find \( y \in \ell^p \) such that

\[ d(x,y) < \epsilon. \]  

This \( \ell \) is dense in \( \ell^p \) and since it is countable, \( \ell^p \) is separable.

1) Consider \( x = (1,1,0,\ldots) \in \ell^p \) and \( y = (1,-1,0,\ldots) \in \ell^p \)

\[ \|x\|_p = \|y\|_p = 2^{1/p} \quad \text{and} \quad \|x+y\| = \|x-y\| = 2. \]

But \( \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \iff 2^{2/p} = 1 \iff p = 2. \]

2) Consider \( x_m(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2} \\ mt - \frac{w}{2} & \frac{1}{2} \leq t \leq \frac{1}{2} + \frac{1}{w} \\ 1 & \frac{1}{2} + \frac{1}{w} \leq t \leq 1 \end{cases} \)

which is continuous.

Assume \( x(t) \) continuous such that \( \|X_n - x\|_{L^2} \to 0 \) then

\[ \int_0^{1/2} |x(t)|^2 \, dt + \int_{1/2}^{1/2 + 1/w} |x_n(t) - x(t)|^2 \, dt + \int_{1/2 + 1/w}^1 |1 - x(t)|^2 \, dt \to 0 \]

and the \( x(t) = 0 \) for \( 0 \leq t < \frac{1}{2} \) but \( x(t) = 1 \) for \( \frac{1}{2} < t \leq 1 \)

and \( x(t) \) is not continuous.

But \( x_m \) is Cauchy if one attempts the unpleasant calculation for \( n > w \)

\[ \|x_n - x_m\|_{L^2}^2 = \int_{1/2}^{1/2 + 1/w} \left( wt - \frac{w}{2} - nt + \frac{n}{2} \right)^2 \, dt + \int_{1/2 + 1/w}^{1/2 + 1/w} \left( 1 - wt + \frac{w}{2} \right)^2 \, dt \]

\[ = \frac{(m-n)^2}{3n^2} < \frac{1}{3m} - \frac{1}{3n} \]