Review for exam 1 Math 442

Coverage: Sections 1.1-1.6. Sections 2.1-2.5. Sections 3.1-3.5 (exclude pages 63-65, the finite interval case.)

1) Review HW sets 1-4.

2) Consider a function \( u: \mathbb{R}^2 \rightarrow \mathbb{R} \). Solve the following PDEs

i) \( \frac{\partial^2 u(x,y)}{\partial x^2} = 0 \)

ii) \( \frac{\partial^2 u(x,y)}{\partial x^2} + u(x,y) = 0 \)

iii) \( \frac{\partial^2 u(x,y)}{\partial x \partial y} = 0 \)

iv) \( \frac{\partial u}{\partial t} + \nabla \cdot (\nabla u) = 0 \)

\( u(x,0) = f(x) \)
\( v) \quad a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} = 0, \quad a, b \text{ constants} \)

\( vi) \quad \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \)

Ans: \( u(x, y) = f(e^{-x}y) \), \( f \) arbitrary

\( vii) \quad \frac{\partial u}{\partial x} + 2xy \frac{\partial u}{\partial y} = 0 \)

Ans: \( u(x, y) = f(x^2 + \frac{1}{y}) \), \( f \) arbitrary

\( viii) \quad u_t + cu_x = g(x, t) \)

\( u(x, 0) = f(x) \)

Ans: \( u(x, t) = f(x - ct) + \int_{-t}^{t} g(x + (s-t)c, s) \, ds \)

Hint: For problems \( iv) - viii) \) you should apply the method of characteristics.
3) Consider \( u''(x) + u'(x) = f(x), \ 0 < x < l \)

\[
u'(0) = u(0) = \frac{1}{2} \left[ u'(l) + u(l) \right]
\]

a) Is the solution unique?

Ans: No, but you should explain.

b) Is there a condition that \( f(x) \) must satisfy for existence?

Hint: Integrate the equation from 0 to \( l \).

4) Reduce the elliptic equation

\[
\begin{align*}
x_{xx} + 3y_{yy} - 2x_{x} + 24y_{y} + 5u & = 0 \text{ into} \\
x_{xx} + 3y_{yy} + cy = 0 \text{ by the change} \\
u = ve^{ax+by}
\end{align*}
\]

Hint: Pick \( a \) and \( b \).

Can you further reduce the equation
to \( V_{xx} + V_{yy} + CV = 0 \)?

Hint: Change the scale \( y' = y \).

5) Solve the initial value problem

\[
\begin{align*}
\Phi_{tt} - c^2 \Phi_{xx} &= 0, & -\infty < x < \infty \\
\Phi(x,0) &= \psi(x), & \Phi_t(x,0) = \Psi(x)
\end{align*}
\]

6) Consider the quantity

\[
E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (\rho \Phi_t^2 + T \Phi_x^2) \, dx \quad \text{Where}
\]

\[
c^2 = \frac{1}{\rho} \quad \text{Show that if } \Phi \text{ solves (*) then}
\]

\[
E(t) = E(0). \quad \text{What is the assumption that we use on the data } \psi \text{ and } \Psi \text{ to solve this problem? Why the finite speed of}
\]
propagation principle is helpful?

7) State the maximum principle for the diffusion equation
   \[ u_t = ku_{xx}, \quad 0 \leq x \leq l, \quad 0 \leq t \leq T \]
on a rectangle.

8) a) Use the maximum principle to prove uniqueness for
   \[ u_t - ku_{xx} = f(x, t) \]
   \[ u(x, 0) = \varphi(x) \]
   \[ u(0, t) = g(t), \quad u(l, t) = h(t) \]

b) Use the energy
   \[ E(t) = \frac{1}{2} \int_0^l \left[ u(x, t) \right]^2 dx \]
to prove uniqueness.

c) Use the maximum principle to prove stability of
\[ u_t - ku_{xx} = 0 \]
\[ u(x,0) = \varphi(x) \quad 0 < x < l, \quad t > 0 \]
\[ u(0,t) = u(l,t) = 0 \]

9) Show that \[ \int_{-\infty}^{\infty} e^{-t^2} \, dt = \sqrt{\pi} \]

10) i) By using the substitution

\[ \varphi(x,t) = \varphi \left( \frac{x}{\sqrt{4kt}} \right) \text{ solve } \varphi_t = k\varphi_{xx} \]

with

\[ \lim_{t \to 0} \varphi(x,t) = 1 \text{ for } x > 0 \text{ and } t > 0 \]
\[ \lim_{t \to 0} \varphi(x,t) = 0 \text{ for } x < 0. \]

ii) Show that \[ S(x,t) = \frac{\varphi}{n_x} (x/t) \] solves the diffusion equation in 1D, \[ u_t = ku_{xx}. \]

iii) Show that for \( t > 0 \),
\[ \int_{-\infty}^{\infty} S(x-y,t) \varphi(y) dy \text{ also solves the } \]
\[ \text{diffusion equation in } 1d. \]

iv) Show that \[ \int_{-\infty}^{\infty} S(t,x) dx = 1 \] and that
\[ \text{for any fixed } \delta > 0, \max_{1 \times 1 > \delta} S(t,x) \to 0 \text{ as } t \to 0. \]

v) Let \( \varphi \) be a bounded and continuous function. Show that if \( u(x,t) = \int_{-\infty}^{\infty} S(x-y,t) \varphi(y) dy \) then
\[ \lim_{t \downarrow 0} u(x,t) = \varphi(x). \]

vi) Solve a) \[ V_t - kV_{xx} = 0, \quad 0 < x < \infty, \quad t > 0 \]
\[ V(x,0) = \varphi(x) \]
\[ V(0,t) = 0 \]
b) \[ W_t - kW_{xx} = 0, \quad 0 < x < \infty, \quad t > 0 \]
w(x, 0) = \psi(x), \quad W_x(0, t) = 0. \quad (8)

12) Solve \quad u_t - ku_{xx} = f(x, t) \quad -\infty < x < \infty
\quad u(x, 0) = \psi(x) \quad t > 0

13) Solve \quad u_t - ku_{xx} = f(x, t) \quad 0 < x < \infty, \quad t > 0
\quad u(x, 0) = \psi(x)
\quad u(0, t) = 0

14) Solve \quad u_t - ku_{xx} = f(x, t) \quad 0 < x < \infty, \quad t > 0
\quad u(x, 0) = \psi(x)
\quad u(0, t) = h(t)

Hint: Set \quad v(x, t) = u(x, t) - h(t) \quad and
reduce to problem 13.

15) i) Solve \quad v_{tt} - c^2 v_{xx} = 0
\quad 0 < x < \infty \quad v(x, 0) = \psi(x), \quad v_t(x, 0) = \psi(x)
\quad -\infty < t < \infty \quad v(0, t) = 0

When in addition
1) \( x > c|t| \) and ii) \( x < c|t| \).

\[
\text{Ans: i) } v(x,t) = \frac{1}{2} \left[ \varphi(x+ct) + \varphi(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) \, dy
\]

\[
\text{ii) } v(x,t) = \frac{1}{2} \left[ \varphi(x+ct) - \varphi(ct-x) \right] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(y) \, dy
\]

Answer to 3) a) Let \( u \) and \( v \) solution. Set \( w = u - v \). Then \( w'' + w' = 0 \) or

\[
w(x) = C_1 e^{-x} + C_2 \Rightarrow u(x) = v(x) + C_1 e^{-x} + C_2
\]

Then \( u'(x) = v'(x) - C_1 e^{-x} \) and \( u(0) = u'(0) \)

\( v(0) = v'(0) \) if \( 2C_1 + C_2 = 0 \). Thus picking \( C_2 = 1 \) and \( C_1 = -\frac{1}{2} \), given \( u(x) \) a solution then \( v(x) = u(x) + \frac{1}{2} e^{-x} - 1 \) is another solution of \( v'' + v' = f \), \( v(0) = v'(0) = \frac{1}{2} \left[ v(1) + v'(1) \right] \)

16) Solve #4 Section 3.4.
17) Solve #1 Section 3.3