Problem 1 (20 points)

Assume $u(x,y)$ is a function of two variables. Solve the equation

$$u_x + 2xy^2u_y = 0.$$

First solve

$$\frac{dy}{dx} = 2xy^2.$$ Find the solution curve $y = y(x)$ and check that

$$\frac{du}{dx} = \frac{u_x}{x} + \frac{u_y}{y} = \frac{u_x}{x} + u_y \cdot 2xy^2 = 0.$$

Thus

$$u(x, y(x)) = f(C) \quad \text{for } C \text{ arbitrary. What is } C?$$

$$\frac{dy}{dx} = 2xy^2 \Rightarrow \frac{dy}{y^2} = 2x \, dx \Rightarrow -\frac{1}{y^2} = x + C \Rightarrow y = \left(x + C\right)^{-1}$$

or

$$C = -x - \frac{1}{y}.$$ Thus

$$u(x, y) = f(-x - \frac{1}{y}) \quad \text{or} \quad f\left(x^2 + \frac{1}{y}\right).$$
Problem 2 (20 points)

Consider the problem

$$u''(x) + u'(x) = f(x)$$

valid for $0 \leq x \leq l$, where $f(x)$ is a given function and the boundary conditions are given by

$$u'(0) = u(0) = \frac{1}{2}[u'(l) + u(l)].$$

Is there a condition that $f(x)$ must satisfy in order for the equation to have a solution?

Integrate

$$u''(x) + u'(x) = f(x) \quad \text{from} \quad 0 \quad \text{to} \quad l.$$

$$\int_0^l [u''(x) + u'(x)] \, dx = \int_0^l f(x) \, dx$$

$$u'(l) + u(l) - u'(0) - u(0) = \int_0^l f(x) \, dx$$

But

$$u(0) = \frac{1}{2} \left[ u'(0) + u(0) \right]$$

$$u'(0) = \frac{1}{2} \left[ u'(0) + u(0) \right] \quad \implies \quad u(0) + u'(0) = u'(0) + u(0)$$

Thus

$$\int_0^l f(x) \, dx = 0.$$
Problem 3 (20 points)

Consider the inhomogeneous heat equation for $0 < x < l$, $t > 0$, $k > 0$.

$$u_t - ku_{xx} = f(x, t),$$
$$u(x, 0) = \phi(x), \quad u(0, t) = g(t), \quad u(l, t) = h(t). \tag{1}$$

Use the energy $E(u)(t) = \frac{1}{2} \int_0^l u^2(x, t)dx$ to prove uniqueness.

Assume $V$ and $u$ solve (1)-(2). Then

$W = u - V$ solves

$$W_t - kW_{xx} = 0$$

Now

$$\frac{dE}{dt} = \frac{d}{dt} \frac{1}{2} \int_0^l w^2(x, t)dx$$

$$= \int_0^l W W_t dx = \int_0^l W KW_{xx} dx = k \left. W_x \right|_0^l - k \int_0^l W_x^2 dx$$

$$= -k \int_0^l W_x^2 dx \leq 0$$

Thus since $\frac{dE}{dt} \leq 0$, $E(t) \leq E(0)$. But $E(0) = 0$ since $W(x, 0) = 0$.

Thus $E(t) \leq 0$ and since $E(t) \geq 0$ we have $E(t) = 0$.

But

$$\int_0^l W^2 dx = 0 \Rightarrow W = 0 \Rightarrow u = V \text{ uniqueness}$$
Problem 4 (20 points)

Solve the wave equation on the half-line in the case that $0 < x < c t$ and $t > 0$,

\begin{align*}
    u_{tt} - c^2 u_{xx} &= 0, \\
    u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x), \\
    u(0, t) &= 0.
\end{align*}

Define

\begin{align*}
    \psi_{odd}(x) &= \begin{cases} 
        \phi(x), & x > 0 \\
        -\phi(-x), & x < 0 \\
        0, & x = 0
    \end{cases} \\
    \psi_{even}(x) &= \begin{cases} 
        \phi(x), & x > 0 \\
        -\phi(-x), & x < 0 \\
        0, & x = 0
    \end{cases}
\end{align*}

Then

\[ u(x, t) = \frac{1}{2c} \left[ \int_{-c-t}^{c-t} \psi_{odd}(x+ct) \, dy + \int_{-c-t}^{c-t} \psi_{odd}(x-ct) \, dy \right] + \int_{x-ct}^{x+ct} \psi_{even}(y) \, dy \]

is odd and $u(0, t) = 0$.

The solution of our problem is the restriction of $u(x, t)$ on $x > 0$.

But when $0 < x < ct$ we have

\begin{align*}
    &\begin{cases} 
        x - ct < 0 \\
        x + ct > 0
    \end{cases} \quad \text{and} \quad \psi_{odd}(x - ct) = -\phi(ct - x) \\
    \psi_{odd}(x + ct) = \phi(ct + x)
\end{align*}

\begin{align*}
    \frac{1}{2c} \int_{-c-t}^{c-t} \psi_{odd}(y) \, dy &= \frac{1}{2c} \int_{-c-t}^{c-t} \psi_{odd}(y) \, dy + \frac{1}{2c} \int_{-c-t}^{c-t} \psi_{even}(y) \, dy \\
    &= -\frac{1}{2c} \int_{-c-t}^{c-t} \psi(-y) \, dy + \frac{1}{2c} \int_{-c-t}^{c-t} \psi(y) \, dy
\end{align*}

\begin{align*}
    s &= -y \\
    dy &= ds \\
    y &= c(t - x) \\
    &= c(t - s)
\end{align*}

\begin{align*}
    \frac{1}{2c} \int_{c(t-x)}^{c(t+x)} \phi(y) \, dy &= \frac{1}{2c} \int_{c(t-x)}^{c(t+x)} \psi(y) \, dy + \frac{1}{2c} \int_{c(t-x)}^{c(t+x)} \phi(y) \, dy
\end{align*}

The solution is $(\text{for } x > 0)$

\[ u(x, t) = \frac{1}{2} \left[ \phi(ct + x) - \phi(ct - x) \right] + \frac{1}{2c} \int^{ct + x}_{ct - x} \psi(y) \, dy \]
Problem 5 (20 points)

a) Solve the initial value problem

\[ u_t + cu_x = 0, \quad (-\infty < x < \infty, \quad t > 0), \]  
\[ u(x, 0) = g(x). \]  

b) Solve the inhomogeneous initial value problem

\[ u_t + cu_x = f(x, t), \quad (-\infty < x < \infty, \quad t > 0), \]  
\[ u(x, 0) = g(x). \]  

\[ a) \quad u_t + cu_x = 0 \]
\[ u(x, 0) = g(x) \]

\[ \frac{dx}{dt} = c \quad \text{then} \quad \frac{du}{dt} = u_x + cu_x = 0 \quad \text{and} \]

\[ u(x, t) = u(x_0, 0) = g(x_0) \]  
\[ B u \quad \frac{dx}{dt} = c \quad \Rightarrow \quad x = ct + A \quad \Rightarrow \]

\[ x(0) = A \]

\[ b) \quad \frac{dx}{dt} = c \quad \text{then} \quad \frac{du}{dt} = f(t) \quad \text{and then} \]

\[ u(x, t) - u(x_0, 0) = \int_0^t \frac{du}{ds} ds = \int_0^t f(x(s), s) ds \]  
As before  
\[ x = ct + A \]
\[ x(s) = cs + A, \quad u(x_0, 0) = g(A) = g(x - ct) \]

and  
\[ u(x, t) = g(x - ct) + \int_0^t f(x + cs - t, s) ds = \]

\[ u(x, t) = g(x - ct) + \int_0^t f(x + cs - t, s) ds \]