Problem 1 (30 points)

a) Solve the differential equation

\[(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.\]

b) A tank initially contains 120 \(L\) of pure water. A mixture containing a concentration of \(\gamma \frac{t^2}{L}\) of salt enters the tank at a rate of \(2\frac{L}{\text{min}}\), and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of \(\gamma\) for the amount of salt in the tank at any time \(t\). Also find the limiting amount of salt in the tank as \(t \to \infty\).

\(a) \quad M(x,y) = y \cos x + 2xe^y, \quad \frac{\partial M}{\partial y} = \cos x + 2xe^y\)

\(N(x,y) = \sin x + x^2e^y - 1, \quad \frac{\partial N}{\partial x} = \cos x + 2xe^y\)

\[\overline{\text{THL}} \quad \psi_x = M \Rightarrow \psi(x,y) = \frac{y \sin x + x^2e^y + C(y)}{2} \]

\[\phi_y = N \quad \text{and} \quad \psi_y = \sin x + x^2e^y + C'(y) = N = \sqrt{\sin x + x^2e^y - 1}\]

\[C(y) = -y + C\]

\[\text{The} \quad y \sin x + x^2e^y - y = C\]

\(b) \quad \text{Rate in} = \gamma \frac{t^2}{L} \times 2 \frac{L}{\text{min}} = 2 \gamma \frac{\frac{t^2}{L}}{\text{min}}\]

\[\text{Rate out} = 2 \frac{L}{\text{min}} \times \frac{\frac{C}{120L}}{\text{min}} Q \frac{g^f}{L} = \frac{Q}{60} y^f_{\text{min}}\]

\[Q(0) = 0 \quad \text{and} \quad C(t) = e^{-\frac{t}{60}}\]

\[Q(t) = C(t) e^{-\frac{t}{60}} + 120g\]

\[Q(0) = 0 \quad \Rightarrow \quad Q(t) = 120g \left(1 - e^{-\frac{t}{60}}\right)\]

\[\lim_{t \to \infty} Q(t) = 120g \; \text{gr.} \]
Problem 2 (20 points)

Solve the initial value problem and determine the interval in which the solution is valid,
\[
\frac{dy}{dx} = \frac{3x}{1 + 4y}, \quad y(1) = 0.
\]
\[
dy(1+4y) = 3x \, dx
\]
\[
y + \frac{y^2}{2} = \frac{3}{2} x + C \quad \quad C = -\frac{3}{2}
\]
\[
2y^2 + y = \frac{3}{2} (x^2 - 1) \quad \text{or} \quad 2y^2 + y - \frac{3}{2} (x^2 - 1) = 0
\]
\[
y_{1,2} = -1 \pm \frac{1 + 12\cdot \frac{3}{2}(x^2 - 1)}{4}
\]
\[
y_{1,2} = -1 \pm \frac{1 + 12(3x^2 - 1)}{4}
\]
\[
y = -\frac{1}{4} \pm \frac{\sqrt{12x^2 - 11}}{4}
\]
Only \( y = -\frac{1}{4} + \frac{\sqrt{12x^2 - 11}}{4} \) satisfies \( y(1) = 0 \).

We need \( 12x^2 - 11 > 0 \) \( \Rightarrow \) \( |x| > \sqrt{\frac{11}{12}} \)
\( x = \frac{\sqrt{11}}{12} \) given \( y = -\frac{1}{4} \) and \( \frac{dy}{dx} \) is infinite.
\( (-\infty, -\frac{\sqrt{11}}{12}) \cup (\frac{\sqrt{11}}{12}, \infty) \) the initial condition is
\( \text{in } (\frac{\sqrt{11}}{12}, \infty) \) and thus \( (\frac{\sqrt{11}}{12}, \infty) \) is the interval of existence.
Problem 3 (20 points)

Solve the differential equation
\[
\frac{dy}{dt} = \epsilon y - \sigma y^3
\]

for \(\epsilon > 0\) and \(\sigma > 0\).

\(y = 0\) is a solution. Let \(u = y^{-2}\) 

\[
\frac{du}{dt} = -2 y^{-3} \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = -\frac{1}{2} y^{-3} \frac{du}{dt}
\]

\[
-\frac{1}{2} y^{-3} \frac{du}{dt} = \epsilon y - \sigma y^3 \Rightarrow -\frac{1}{2} y^{-2} \frac{du}{dt} = \epsilon - 6 y^2
\]

or

\[
-\frac{1}{2} \frac{du}{dt} = \epsilon u - 6 \quad \text{or} \quad \frac{du}{dt} + 2 \epsilon u u = 2 \epsilon
\]

\(u(t) = Ce^{-2 \epsilon t}\) with \(c'(0) e^{-2 \epsilon t} = 2 \epsilon \Rightarrow c'(0) = 2 \epsilon e^{2 \epsilon t}

\(c(t) = Ce^{-2 \epsilon t} + \frac{6}{\epsilon}\) and \(u = \frac{-2 \epsilon t}{Ce^{-2 \epsilon t} + \epsilon}\)

\[
y = \frac{1}{\sqrt{Ce^{-2 \epsilon t} + \frac{6}{\epsilon}}} = \frac{\sqrt{\epsilon}}{\sqrt{6 + Ce^{-2 \epsilon t}}}
\]
Problem 4 (20 points)

Consider the differential equation \( \frac{dy}{dt} = y^2(y-3) \), \( -\infty < y_0 < \infty \). Sketch the graph of \( f(y) = y^2(y-3) \) versus \( y \) and determine the critical points. Then sketch several graphs of solutions in the \( ty \)-plane.

\[
\begin{align*}
f(y) &= y^2(y-3) \\
f'(y) &= 2y(y-3) + y^2 = y \left[ 2y + y(y-3) \right] = y \left[ 3y - 6 \right] = 3y(y-2) \\
f''(y) &= 3(y-2) + 3y = 3 \cdot 2(y-1) = 6(y-1)
\end{align*}
\]

\[
f'(y) = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = 2
\]

\[
f''(0) = -6 \quad \text{local max} \quad f(0) = 0
\]

\[
f''(2) = 6 \quad \text{local min} \quad f(2) = -4
\]

\[
\frac{dy}{dt} = y^2(y-3)
\]

\[
\frac{d^2y}{dt^2} = f'(y) = y^2(y-3) \cdot 3y(y-2) = 3y^3(y-2)(y-3)
\]

\[
\begin{array}{c|c|c|c}
& 0 & 2 & 3 \\
\hline
\text{concave down} & - & + & + \\
\text{concave up} & + & - & + \\
\text{decreasing} & - & + & + \\
\text{increasing} & - & - & - \\
\end{array}
\]
Problem 5 (20 points)

Show that if
\[
\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x} = R,
\]
where \( R \) depends on the quantity \( xy \) only, then the differential equation
\[
M + N \frac{dy}{dx} = 0
\]
has an integrating factor of the form \( \mu(xy) \). Find a general formula for this integrating factor.

\[
\mu + N \frac{dy}{dx} = 0 \quad \text{multiply by } \mu(x,y)
\]

\[
\mu M + \mu N \frac{dy}{dx} = 0
\]

We need \( (\mu M)_x = (\mu N)_y \) by Cem

\[
\mu_x M + \mu_y N = \mu_x N + \mu_x N
\]

But \( \mu_y = \mu' x \), \( \mu_x = \mu' y \)

or \( (N_x - M_y) \mu = (x N - y N) \mu' \) or \( R \mu = \mu' \)

\[
\frac{d\mu}{dt} = R(t) \mu(t) \implies \mu(t) = e^{\int R(t) dt} \quad \text{where } \quad t = xy
\]