"DISTORTION OF DIMENSION BY SOBOLEV AND QUASICONFORMAL MAPPINGS" EIGHTH SCHOOL ON ANALYSIS IN METRIC SPACES LEVICO TERME, ITALY JUNE 16–18, 2014

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Reference List

Quasiconformal mappings in $\mathbb{R}^n$ and the Heisenberg group.

(1) J. Väisälä, *Lectures on n-dimensional quasiconformal mappings.* Lecture Notes in Mathematics, vol. **229**. Springer-Verlag, Berlin/New York, 1971.

The standard reference for the geometric, metric and analytic theory of quasiconformal mappings in $\mathbb{R}^n$. Very readable and still extremely relevant.

(2) F. W. Gehring, "The L^p-integrability of the partial derivatives of a quasiconformal mapping", *Acta Math.* **130** (1973), 265–277.

Fred Gehring's signature result on the higher Sobolev integrability of quasiconformal mappings: *each K-quasiconformal map of domains in $\mathbb{R}^n$, $n \geq 2$, lies in $W^{1,p}_{loc}$ for all $p < p(n,K)$, where $p(n,K) > n$.* This was previously known in dimension two only (a result of Bojarski).

(3) F. W. Gehring and J. Väisälä, "Hausdorff dimension and quasiconformal mappings", *J. London Math. Soc. (2)* (1973), no. 6, 504–512.

The paper which initiated the line of research described in this lecture series. Gehring and Väisälä give dilatation-dependent bounds on the distortion of the Hausdorff dimension of subsets of $\mathbb{R}^n$ by quasiconformal mappings. Even though both this paper and the preceding one were both published in 1973, this paper was completed before the results of the previous one were available, hence, the Hausdorff dimension distortion theorems in this paper are only stated in dimension two. However, the proofs work in any dimension once higher integrability is established.

(4) P. Tukia and J. Väisälä, "Quasiconformal extension from dimension n to $n+1$", *Ann. of Math. (2)* **115** (1982), no. 2, 331–348.

The main result of this paper is the following: *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a quasiconformal mapping, $n \geq 2$. Then there exists a quasiconformal mapping $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ s.t. $F|_{\mathbb{R}^n} = f$.* Here $\mathbb{R}^n$ is identified with the subspace $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. This result was previously known for low values of n, but the proofs in those cases rely on specific features of low-dimensional Euclidean spaces.

(5) A. Korányi and H. M. Reimann, “Quasiconformal mappings on the Heisenberg group”, *Invent. Math.*, **80** (1985), no. 2, 309–338.

(6) A. Korányi and H. M. Reimann, “Foundations for the theory of quasiconformal mappings on the Heisenberg group”, *Adv. Math.*, **111** (1995), no. 1, 1–87.

These two papers present a detailed account of the analytic, metric and geometric theory of quasiconformal mappings on the sub-Riemannian Heisenberg group. Topics addressed include: equivalence of definitions of quasiconformal mappings, Beltrami equations, quasiconformal flows, Liouville's theorem on the rigidity of 1-quasiconformal mappings, and the interplay between the quasiconformal geometry of the (compactified) Heisenberg group and complex hyperbolic geometry.

(7) T. Iwaniec, “The Gehring lemma” in *Quasiconformal mappings and analysis (Ann Arbor, MI, 1995)*, 181-204, Springer, New York, 1998.

A well-written and engaging overview of Gehring's lemma on reverse Hölder inequalities and its impact on PDE and analysis during the time period 1973–1995. Highly recommended.

(8) C. J. Bishop, “Quasiconformal mappings which increase dimension”, *Ann. Acad. Sci. Fenn. Math.*, **24** (1999), no. 2, 397–407.

Bishop proves that there is no obstruction to raising the dimension of subsets of $\mathbb{R}^n$ by a quasiconformal mapping, except for the obvious one imposed by the Hölder continuity of such maps. More precisely, he shows that to any set $E \subset \mathbb{R}^n$ of positive dimension and any $\epsilon > 0$ there exists a quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\dim f(E) > n - \epsilon$.

Conformal dimension and global quasiconformal dimension.

(1) P. Pansu, “Dimension conforme et sphère à l'infini des variétés à courbure négative” [Conformal dimension and sphere at infinity of manifolds of negative curvature], *Ann. Acad. Sci. Fenn. Ser. A I Math.* **14** (1989), no. 2, 177–212.

In this groundbreaking paper the author introduces the notion of conformal dimension and computes the conformal dimension of the boundaries of the classical rank one symmetric spaces.

(2) M. Bourdon, “Au bord de certains polyèdres hyperboliques”, *Ann. Inst. Fourier (Grenoble)*, **45** (1995), no. 1, 119–141.

The author computes the conformal dimension of certain nonsmooth metric spaces arising as the boundaries at infinity of Gromov hyperbolic spaces (hyperbolic buildings). In connection with this lecture series, this paper is of interest for its introduction of an abstract metric space criterion for conformal dimension lower bounds. (A similar abstract criterion appears in the previous paper of Pansu.)

(3) J. T. Tyson, “Sets of minimal Hausdorff dimension for quasiconformal maps”, *Proc. Amer. Math. Soc.*, **128** (2000), no. 11, 3361–3367.

The main result of the paper provides conditions on a metric space sufficient to conclude minimality for conformal dimension. The conditions are phrased in terms of the modulus of curve families. As a corollary it is shown that all Ahlfors Q-regular Q-Loewner spaces ($Q > 1$) are minimal for conformal dimension. It is conjectured that no spaces with conformal dimension strictly between zero and one exist. (This conjecture was later proved by Kovalev; see the reference below.)

(4) J. T. Tyson, "Lowering the Assouad dimension by quasisymmetric mappings", *Illinois J. Math.*, **45** (2001), no. 2, 641–656.

A variant form of the above conjecture is established for the Assouad dimension. Two results are proved: any metric space with Assouad dimension strictly less than one has conformal Assouad dimension zero, and any subset of Euclidean space with Assouad dimension strictly less than one has global quasiconformal Assouad dimension zero.

(5) Z. M. Balogh, "Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group", *J. Anal. Math.*, **83** (2001), 289–312.

Conformal dimension and global quasiconformal dimension of subsets of the Heisenberg group are considered. By considering suitable Cantor sets, the author shows that Hausdorff dimension of certain subsets can be distorted at will by quasiconformal mappings. He also constructs specific sets (in dimension at least one) whose Hausdorff dimension cannot be reduced by a quasiconformal mapping. New techniques, especially the Korányi–Reimann technique of quasiconformal flows, are required for the construction of the relevant quasiconformal mappings.

(6) M. Bourdon and H. Pajot, "Cohomologie ℓ_p et espaces de Besov", *J. Reine Angew. Math.*, **558** (2003), 85–108.

The authors provide a framework for interpreting results of analysis on metric spaces in terms of the geometry of certain hyperbolic fillings. ℓ_p cohomology of hyperbolic spaces is studied and related to the quasisymmetric geometry of the boundary. The Ahlfors regular conformal dimension of the boundary is related to a critical exponent for the ℓ_p cohomology, and also to certain Besov spaces.

(7) S. Keith and T. Laakso, "Conformal Assouad dimension and modulus", *Geom. Funct. Anal.* **14** (2004), no. 6, 1278–1321.

An extremely influential paper whose main result has been used effectively in applications to geometric group theory and rigidity. The authors give a characterization of spaces minimal for the Ahlfors regular conformal dimension, in terms of nontriviality of moduli of curve families in weak tangent spaces. Warning: the paper is long and quite technical. A new proof of the main result of this paper has recently been given by Carrasco Piaggio (see below).

(8) M. Bonk and B. Kleiner, "Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary", *Geom. Topol.* **9** (2005), 219–246.

The authors provide a criterion, based on conformal dimension, which guarantees that if the Gromov boundary of a Gromov hyperbolic group is assumed a priori to be a topological 2-sphere, then indeed it is a quasisymmetric 2-sphere. This resolves Cannon's conjecture under such an additional assumption.

(9) L. V. Kovalev, "Conformal dimension does not assume values between zero and one", *Duke Math. J.*, **134** (2006), no. 1, 1–13.

Kovalev proves the nonexistence of spaces with conformal dimension strictly between zero and one. Put another way, for any metric space X with $\dim X < 1$ and any $\epsilon > 0$, there exists a quasisymmetric map $f : X \to Y$ of X onto another metric space Y so that $\dim Y < \epsilon$. The map in question is constructed via techniques from convex geometry, and the proof establishes the following stronger fact: *for any real separable Banach space V, any subset $E \subset V$ with* $\dim E < 1$, *and any $\epsilon > 0$ there exists a quasisymmetric homeomorphism $f : V \to V$ so that* $\dim f(E) < \epsilon$.

(10) J. T. Tyson, "Global conformal Assouad dimension in the Heisenberg group", *Conform. Geom. Dyn.*, **12** (2008), 32–57.

This paper is a counterpart of the previous paper "Lowering the Assouad dimension by quasisymmetric mappings" (*Illinois J. Math.*, **45**, 2001) by the same author. The main result is that global quasiconformal Assouad dimension takes on no values between zero and one in the sub-Riemannian Heisenberg group $\mathbb{H}^n$. The proofs use quasiconformal flows in the sense of Korányi–Reimann and an analysis of the sub-Riemannian tubular neighborhoods of smooth noncharacteristic hypersurfaces.

(11) J. M. Mackay, "Spaces and groups with conformal dimension greater than one", *Duke Math. J.* **153** (2010), no. 2, 211–227.

The main result of this paper gives geometric criteria sufficient to conclude that the conformal dimension is bounded below by a constant strictly bigger than one. The criteria ensure that the space in question contains a rough (deformed) copy of a product set $E \times [0,1]$, where E is a Cantor set. Such sets are already known to have conformal dimension bounded below by $1+\epsilon$ for suitable $\epsilon > 0$. The result has applications to the algebraic study of finitely generated infinite groups.

(12) J. M. Mackay and J. T. Tyson, *Conformal dimension: theory and application*, University Lecture Series, AMS, 2010.

A general survey and overview of conformal dimension, emphasizing both the internal theory (as a subject in its own right) as well as applications to geometric group theory, dynamics and rigidity.

(13) M. Carrasco Piaggio, "On the conformal gauge of a compact metric space", *Ann. Sci. Éc. Norm. Supér. (4)* **46** (2013), no. 3, 495548.

The author considers the conformal Assouad dimension (also known as the Ahlfors regular conformal dimension) of compact metric spaces. He obtains a combinatorial description all of quasisymmetrically equivalent metrics and uses this to compute the Ahlfors regular conformal dimension in terms of a critical exponent for combinatorial moduli. Using his methods, he also provides an independent proof of the main result of the paper of Keith and Laakso cited above.

Sobolev mappings from domains in $\mathbb{R}^n$ and the Heisenberg group.

(1) R. P. Kaufman, "Sobolev spaces, dimension, and random series", *Proc. Amer. Math. Soc.* **128** (2000), no. 2, 427–431.

The author discusses the distortion of dimensions of subsets of $\mathbb{R}^n$ under Sobolev mappings. After briefly recalling the Morrey–Sobolev inequality and its implications for dimension distortion bounds, the author gives a detailed proof of the following complementary result: *Let E be a compact subset of $\mathbb{R}^n$ such that $\mathcal{H}^s(E) > 0$. Then there exists a $W^{1,p}$ mapping f defined on $\mathbb{R}^n$ such that* $\dim f(E) \geq \frac{ps}{p-n+s}$. Similar results are also obtained for packing dimension.

(2) S. Hencl and P. Honzík, "Dimension of images of subspaces under Sobolev mappings", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **29** (2012), no. 3, 401–411.

Some of the results in the following paper are extended to the case of subcritical Sobolev mappings $f \in W^{1,p}$, $p \leq n$, for p-quasicontinuous representatives.

(3) Z. M. Balogh, R. Monti and J. T. Tyson, "Frequency of Sobolev and quasiconformal dimension distortion", *J. Math. Pures Appl. (9)*, **99** (2013), no. 2, 125–149.

The main result of the paper is a sharp estimate for the size of the exceptional set of fibers of an orthogonal projection map in $\mathbb{R}^n$, such that the dimension of the image of such fiber under a $W^{1,p}$ mapping, $p > n$, exhibits a prespecified dimension increase. Applications for quasiconformal mappings are presented, and examples demonstrating the sharpness of both the Sobolev and quasiconformal results are provided.

(4) Z. M. Balogh, J. T. Tyson and K. Wildrick, "Frequency of Sobolev dimension distortion of horizontal subgroups of Heisenberg groups", preprint.

The results of the previous paper are generalized to the sub-Riemannian Heisenberg group with respect to the foliation by left cosets of a horizontal homogeneous subgroup. The proofs are inspired by Gehring's proof of the ACL property of Euclidean quasiconformal mappings and Mostow's alternative argument for the ACL property of Heisenberg quasiconformal mappings. Other tools from geometric measure theory (e.g., Mattila's slicing theorem) also play a role.

Quasisymmetric and Sobolev maps in metric spaces.

(1) P. Tukia and J. Väisälä, "Quasisymmetric embeddings of metric spaces", *Ann. Acad. Sci. Fenn. Ser. A I Math. 5* (1980), no. 1, 97–114.

Quasisymmetric mappings were 'officially' introduced in this paper (although the concept had existed for some time before that, particularly for mappings of the real line considered as boundary values of quasiconformal mappings of the upper half plane). This paper also contains the quasisymmetric classification of one-dimensional manifolds (the line and the circle). The much more difficult higher-dimensional case remains open, except in dimension two (see the work of Bonk and Kleiner referenced below).

(2) J. Heinonen and P. Koskela, "Definitions of quasiconformality", *Invent. Math.*, **120** (1995), no. 1, 61–79.

(3) J. Heinonen and P. Koskela, "Quasiconformal maps in metric spaces with controlled geometry", *Acta Math.*, **181** (1998), no. 1, 161.

These two papers inaugurated the modern theory of analysis in metric measure spaces. The primary motivating problem is: *when are quasiconformal mappings quasisymmetric?* The authors answer this question in a broad class of metric measure spaces including all sub-Riemannian Carnot groups. The notions of *upper gradient* and the *Poincaré inequality* on a metric measure space, which have become canonical tools in the subject, were first introduced here.

(4) J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson, "Sobolev classes of Banach space valued functions and quasisymmetric maps", *J. d'Analyse Math.*, **85** (2001), 87–139.

In this paper the equivalence of definitions of quasiconformal and quasisymmetric mappings of metric measure spaces is completed by the inclusion of a suitable analytic definition. Such analytic definition relies on the notion of Sobolev mapping between metric spaces, which is here developed from first principles. Such mappings are understood by means of an isometric embedding of the target into a Banach space. The final section of the paper discusses how quasisymmetric mappings affect the Cheeger differential structure on Ahlfors regular metric measure spaces supporting a suitable Poincaré inequality.

(5) Z. M. Balogh, J. T. Tyson and K. Wildrick, "Dimension distortion by Sobolev mappings in foliated metric spaces", *Anal. Geom. Metr. Spaces*, **1** (2013), 232–254.

The results of the earlier paper by Balogh, Monti and Tyson are extended to the metric space framework, for foliations by the level sets of a David–Semmes regular mapping. Under natural metric assumptions (Ahlfors Q-regularity and the validity of a Poincaré inequality), estimates are given for the size of the exceptional set of points for which the fiber of a David–Semmes regular mapping under a $W^{1,p}$ mapping, $p > Q$, exhibits a prespecified dimension increase. Natural examples of this framework include the foliation of the Heisenberg group by cosets of a vertical (normal) homogeneous subgroup, or by right cosets of a horizontal homogeneous subgroup. For left cosets of a horizontal homogeneous subgroup, see the other paper by the same three authors cited above.

Geometric measure theory.

(1) P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics, **44**. Cambridge University Press, Cambridge, 1995.

One of the standard references for Euclidean geometric measure theory, including Hausdorff and Minkowski dimension, rectifiability and Hausdorff dimension estimates under various mappings and transformations.

(2) J. Luukkainen, "Assouad dimension: antifractal metrization, porous sets, and homogeneous measures", *J. Korean Math. Soc.* **35** (1998), no. 1, 23–76.

A comprehensive and highly detailed treatment of Assouad dimension in the metric space setting.