*Distortion of dimension by Sobolev and quasiconformal mappings*

J. Tyson

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**I.** Introduction and overview, quasiconformal maps of $\mathbb{R}^n$ and their effect on Hausdorff dimension
**II.** Global quasiconformal dimension in $\mathbb{R}^n$
**III.** Conformal dimension of metric spaces
**IV.** Sobolev dimension distortion in $\mathbb{R}^n$ and in metric spaces
**V.** QC and Sobolev dimension distortion in the sub-Riemannian Heisenberg group

In this final lecture we will survey known results on distortion of dimension by Sobolev and quasiconformal mappings of the sub-Riemannian Heisenberg group $\mathbb{H}^n$.

We equip $\mathbb{H}^n$ with the Carnot-Carathéodory metric $d_{cc}$ and recall that the Hausdorff dimension of $(\mathbb{H}^n, d_{cc})$ is $Q = 2n + 2$.

A homeomorphism $f : \Omega \to \Omega'$ between domains in $\mathbb{H}^n$ is *(metrically) $K$-quasiconformal* if

$$\limsup_{r\to 0} \frac{\sup\{d_{cc}(f(p), f(q)) \,:\, d_{cc}(p,q) = r\}}{\inf\{d_{cc}(f(p), f(q')) \,:\, d_{cc}(p,q') = r\}} \le K \qquad \forall\, p \in \Omega.$$

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A major difficulty, however, lies in the lack of effective methods for constructing Heisenberg quasiconformal maps. For instance,

- naive versions of the radial stretch map $x \mapsto |x|^{a-1}x$ fail to be quasiconformal (cf. recent work of Balogh–Fässler–Platis),
- PL techniques are not obviously available,
- no theory of Heisenberg quasiuniform convexity exists to date.

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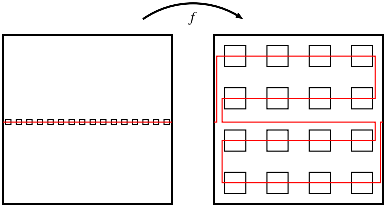

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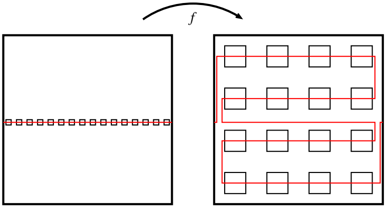


**2.** Gehring–Väisälä theorem: for each $0 < s < Q$ and $K \geq 1$ $\exists\, 0 < \beta \leq \alpha < Q$ s.t. $\beta \leq \dim f(E) \leq \alpha$ whenever $E \subset \mathbb{H}^n$ with $\dim E = s$ and $f$ is $K$-qc.

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For $2n+1 \le s \le Q$, let $E = \bigcup_{r \in C} \partial B_H(o, r)$ be a Cantor set's worth of Korányi spheres $\partial B_H(o, r) = \{p \in \mathbb{H}^n : d_H(o, p) = r\}$. Apply a version of the Bourdon–Pansu criterion for such foliations.

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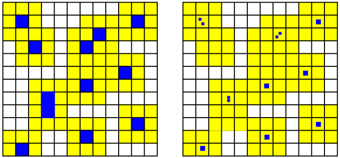
**4.** (T, 2008) Kovalev's theorem holds for the Assouad dimension on $\mathbb{H}^n$.

In other words, if $E \subset \mathbb{H}^n$, $\dim_A E < 1$ then $GQC\dim_{A, \mathbb{H}^n} E = 0$.



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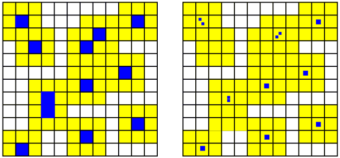

**Question:** Kovalev's theorem in $\mathbb{H}^n$ for Hausdorff dimension?

# Quasiconformal flows

Recall the standard left invariant vector fields $X_1, Y_1, \ldots, X_n, Y_n$ generating $H\mathbb{H}^n$.

$$X_j(p) = \tfrac{\partial}{\partial x_j} + 2y_j \tfrac{\partial}{\partial t}, \qquad Y_j(p) = \tfrac{\partial}{\partial y_j} - 2x_j \tfrac{\partial}{\partial t}$$

Korányi and Reimann (1995) give conditions on a vector field $V$ which guarantee that the time $s$ map $p(0) \mapsto p(s)$ associated to the ODE $\dot{p} = V(p)$ is $K(s)$-qc. Basic structure is

$$V = \varphi\, T + \frac{1}{4} \sum_{j=1}^{n} (X_j \varphi\, Y_j - Y_j \varphi\, X_j) \qquad \text{for } \varphi \in C^\infty(\mathbb{H}^n)$$

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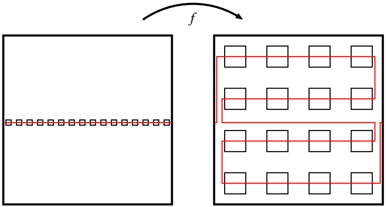
**Examples:** $\varphi = 1$, $\varphi = x_k$, $\varphi = y_k$, $k = 1, \dots, n$ generate left translation flows

$\varphi = 2t \rightsquigarrow V = 2tT + \sum_j (x_j\, X_j + y_j\, Y_j)$ generates dilation flow $p_0 \mapsto \delta_{e^s}(p_0)$




Fix a ball $B(p,R)$ and a family of disjoint balls $B_j = B(p_j, r_j) \subset B(p, r)$.

Choose potentials $\varphi_j$ generating contractive similarities with fixed points at the $p_j$, and define $\varphi = \sum_j \chi_j \varphi_j$ for suitable smooth cutoff functions $\chi_j$.

For suitable $s$, the time $s$ map $F$ associated to $\varphi$ is qc, fixes $\partial B(p, r)$, and contracts each $B_j$. Moreover, $F$ is conformal outside of $B$ and inside each $B_j$. Iterate.

As the flow method only works for smooth maps, use compactness of the space of normalized $K$-qc maps to pass to the limit.

### Theorem (Pansu)

*Every qc map $f$ of domains in $\mathbb{H}^n$ is differentiable almost everywhere in the following sense: for a.e. $p \in \Omega$ the limit*

$$\delta_{1/r} \circ \ell_{f(p)}^{-1} \circ f \circ \ell_p \circ \delta_r$$

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In particular, it follows that $f$ is a.e. diff'ble in horizontal directions, and that $f$ is a *generalized contact map*, $d_h f_p = (Df_p)_*|_{H_p\mathbb{H}^n}$ maps $H_p\mathbb{H}^n$ to $H_{f(p)}\mathbb{H}^n$ a.e.

**Intuition:** The CC ball at $p$ of radius $0 < r \ll 1$ is comparable to the box $[-r,r]^{2n} \times [-r^2, r^2]$, sheared so that the $[-r,r]^{2n}$ factor lies along the subspace $H_p\mathbb{H}^n$. Metric quasiconformality $\Rightarrow$ at a point of differentiability, the size $r$ and size $r^2$ directions cannot be interchanged by $f$.

For $f = (f_1, \ldots, f_{2n+1})$ quasiconformal, we have

$$||d_h f||^Q \leq K \det Df \quad \text{a.e.}$$

In particular, each component $f_j$ lies in the local horiz Sobolev space $W^{1,Q}_{loc}(\Omega)$.

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**Definition**

Given a metric space $Y$, fix an isometric embedding $\kappa$ of $Y$ into a Banach space $V$. Then

$$W^{1,p}(\Omega : Y) := \{ f \in W^{1,p}(\Omega : V) \, : \, f(p) \in \kappa(Y) \text{ a.e.} \}$$

For instance when $Y = \mathbb{H}^n$ we can choose $V = \ell^\infty$ (Fréchet embedding).

Each $f$ qc lies in the metric space-valued Sobolev space $W^{1,Q}_{loc}(\Omega : \mathbb{H}^n)$, in fact, $\exists \, p(\mathbb{H}^n, K) > Q$ s.t. $f$ $K$-qc in $\mathbb{H}^n$ lies in $W^{1,p}_{loc}$ for all $p < p(\mathbb{H}^n, K)$.

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We'll consider the exceptional sets problem for $W^{1,p}$ maps, $p > Q$, for generic elements in foliations of $\mathbb{H}^n$ by left or right cosets of horizontal subgroups.

A smooth submanifold $\Sigma \subset \mathbb{H}^n$ of (topological) dimension $p \leq n$ has CC Hausdorff dimension $p$ if and only if $\Sigma$ is horizontal. Otherwise, it has CC dimension $p+1$. On the other hand, every submanifold $\Sigma$ of topological dimension $> n$ satisfies $\dim \Sigma = p + 1$.

For example, a (smooth) curve $\gamma$ in $\mathbb{H}^1$ is horizontal if and only if $\dim \gamma = 1$, while any surface has CC dimension 3.

Let $V$ be a left invariant horiz vector field, $\mathbb{V} = \{\exp(sV) : s \in \mathbb{R}\}$ the corresponding one-parameter horizontal subgroup.

The Euclidean complement $\mathbb{W}$ of $\mathbb{V}$ is a normal subgroup of $\mathbb{H}^n$, and $\mathbb{H}^n$ admits two semidirect decompositions: $\mathbb{H}^n = \mathbb{W} \ltimes \mathbb{V}$ or $\mathbb{H}^n = \mathbb{V} \rtimes \mathbb{W}$. These decompositions induce projection maps $p^L_{\mathbb{W}} : \mathbb{H}^n \to \mathbb{W}$ and $p^R_{\mathbb{W}} : \mathbb{H}^n \to \mathbb{W}$.

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**Left coset foliation** $\{a * \mathbb{V} = (p^L_{\mathbb{W}})^{-1}(a) \, : \, a \in \mathbb{W}\}$**:**
fibers are all 1-dimensional, $(\mathbb{W}, d_{cc})$ is $(2n+1)$-dimensional,
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**Right coset foliation** $\{\mathbb{V} * a = (p^R_{\mathbb{W}})^{-1}(a) \, : \, a \in \mathbb{W}\}$**:**
fibers are generically 2-dimensional, $\mathbb{W}$ can be equipped with a quotient metric $d$ w.r.t. which $\mathbb{W}$ is $(2n)$-dimensional, $p^R_{\mathbb{W}} : (\mathbb{H}^n, d_{cc}) \to (\mathbb{W}, d)$ is Lipschitz.

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**Lemma:** $p^L_{\mathbb{W}} : (\mathbb{H}^n, d_{cc}) \to (\mathbb{W}, d_E)$ and $p^R_{\mathbb{W}} : (\mathbb{H}^n, d_{cc}) \to (\mathbb{W}, d)$ are both locally David–Semmes 2-regular.

Recall: $\pi : X \to W$ is *locally s-regular* if $\pi$ is locally Lipschitz, onto, and for each compact $K$ $\exists$ $C > 0$ s.t. for every $B_r \subset W$, $\pi^{-1}(B) \cap K$ is covered by $\leq Cr^{-s}$ balls of radius $Cr$.

# Exceptional sets for $\mathbb{H}^n$ Sobolev dimension distortion

joint work with Z. Balogh and K. Wildrick

# Exceptional sets for $\mathbb{H}^n$ Sobolev dimension distortion

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From the metric space results of Lecture IV we obtain conclusions about generic dimension increase for images of cosets of horizontal subgroups. These results are stated for the quotient metric $d$ on $\mathbb{W}$ in the right coset foliation $\mathbb{V} \rtimes \mathbb{W}$ and for the **Euclidean** metric on $\mathbb{W}$ in the left coset foliation $\mathbb{W} \ltimes \mathbb{V}$.

## Corollary

*Let $\Omega \subset \mathbb{H}$ and let $f \in W^{1,p}(\Omega : Y)$, $p > 4$, be continuous. For the horizontal subgroup $\mathbb{V} = \{(x, 0, 0) : x \in \mathbb{R}\}$ with complementary vertical subgroup $\mathbb{W} = \{(0, y, t) : y, t \in \mathbb{R}\}$, and for $2 < \alpha \leq \frac{2p}{p-2}$ we have*

$$\dim_d\{a \in \mathbb{W} : \dim f(\mathbb{V} * a) \geq \alpha\} \leq 2 - p(1 - \tfrac{2}{\alpha}),$$
$$\dim_{d_E}\{a \in \mathbb{W} : \dim f(a * \mathbb{V}) \geq \alpha\} \leq 2 - p(1 - \tfrac{2}{\alpha}).$$

The dimension estimate for left coset foliations on the previous slide is not optimal (since the David–Semmes regularity exponent of the foliation differs from the fiber dimension).

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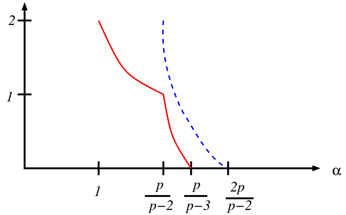
**Theorem**

*Same assumptions as above. For* $1 \le \alpha \le \frac{p}{p-3}$ *we have*

$$\dim_{d_E}\{a \in \mathbb{W} : \dim f(a * \mathbb{V}) \ge \alpha\} \le \begin{cases} 2 - \frac{p}{2}(1 - \frac{1}{\alpha}) & 1 \le \alpha < \frac{p}{p-2} \\ 3 - p(1 - \frac{1}{\alpha}) & \frac{p}{p-2} \le \alpha \le \frac{p}{p-3} \end{cases}$$



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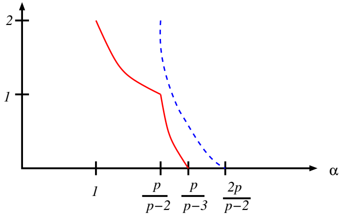


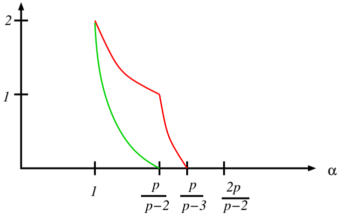


**Example:** For $p > 4$ and $1 < \alpha < \frac{p}{p-2}$, $\exists\, F \subset \mathbb{W}$ compact and $f \in W^{1,p}(\mathbb{H} : \mathbb{R}^2)$ continuous s.t. $\dim f(a * \mathbb{V}) \ge \alpha \;\forall\, a \in F$ and $0 < \mathcal{H}^{2-p(1-1/\alpha)}_{d_E}(F) < \infty$.

For left cosets of $m$-dimensional horizontal subgroups the result reads as follows:

**Theorem**

*Let $\Omega \subset \mathbb{H}^n$ and let $f \in W^{1,p}(\Omega : Y)$, $p > Q$, be continuous. For $m \in \{1, \ldots, n\}$, an $m$-dimensional horizontal subgroup $\mathbb{V}$, and $m \leq \alpha \leq \frac{pm}{p-Q+m}$ we have*

$$\dim_{d_E}\{a : \dim f(a * \mathbb{V}) \geq \alpha\} \leq \begin{cases} (Q-m-1) - \frac{p}{2}(1-\frac{m}{\alpha}) & m \leq \alpha < \frac{pm}{p-2} \\ (Q-m) - p(1-\frac{m}{\alpha}) & \frac{pm}{p-2} \leq \alpha \leq \frac{pm}{p-Q+m} \end{cases}$$

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The estimate recovers the borderline value $Q - m - 1 = 2n + 1 - m = \dim_{d_E} \mathbb{W}$ when $\alpha \to m$, and reaches zero when $\alpha$ tends to the universal upper dimension estimate $\frac{pm}{p-Q+m}$.

The statement is only on the level of dimension, not measure. Also, there is a gap between the positive result and the $\mathbb{H}^1$ example which we are not able to resolve. Such a result would follow, for instance, from a positive answer to the following:

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Again, we used the Eucl metric on $\mathbb{W}$ to estimate the size of the exceptional set.

The estimate recovers the borderline value $Q - m - 1 = 2n + 1 - m = \dim_{d_E} \mathbb{W}$ when $\alpha \to m$, and reaches zero when $\alpha$ tends to the universal upper dimension estimate $\frac{pm}{p-Q+m}$.

The statement is only on the level of dimension, not measure. Also, there is a gap between the positive result and the $\mathbb{H}^1$ example which we are not able to resolve. Such a result would follow, for instance, from a positive answer to the following:

**Question:** Does there exist a locally David–Semmes 1-regular surjection from $\mathbb{H}^n$?

Thanks for your attention!