Distortion of dimension by Sobolev and quasiconformal mappings

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I. Introduction and overview, quasiconformal maps of $\mathbb{R}^n$ and their effect on Hausdorff dimension
II. Global quasiconformal dimension in $\mathbb{R}^n$
III. Conformal dimension of metric spaces
IV. Sobolev dimension distortion in $\mathbb{R}^n$ and in metric spaces
V. QC and Sobolev dimension distortion in the sub-Riemannian Heisenberg group
In this final lecture we will survey known results on distortion of dimension by Sobolev and quasiconformal mappings of the sub-Riemannian Heisenberg group $\mathbb{H}^n$.

We equip $\mathbb{H}^n$ with the Carnot-Carathéodory metric $d_{cc}$ and recall that the Hausdorff dimension of $(\mathbb{H}^n, d_{cc})$ is $Q = 2n + 2$.

A homeomorphism $f : \Omega \to \Omega'$ between domains in $\mathbb{H}^n$ is (metrically) $K$-quasiconformal if

$$\limsup_{r \to 0} \frac{\sup \{d_{cc}(f(p), f(q)) : d_{cc}(p, q) = r\}}{\inf \{d_{cc}(f(p), f(q')) : d_{cc}(p, q') = r\}} \leq K \quad \forall \ p \in \Omega.$$
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A major difficulty, however, lies in the lack of effective methods for constructing Heisenberg quasiconformal maps. For instance,

- naive versions of the radial stretch map \( x \mapsto |x|^{a-1}x \) fail to be quasiconformal (cf. recent work of Balogh–Fässler–Platis),
- PL techniques are not obviously available,
- no theory of Heisenberg quasiuniform convexity exists to date.
Distortion of dimension by quasiconformal maps in $\mathbb{H}^n$

1. For each $0 < s < t < Q$ there exists $E \subset \mathbb{H}^n$ compact and $f : \mathbb{H}^n \to \mathbb{H}^n$ qc s.t. $\dim E = s$ and $\dim f(E) = t$. 
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2. Gehring–Väisälä theorem: for each $0 < s < Q$ and $K \geq 1$ $\exists 0 < \beta \leq \alpha < Q$ s.t. $\beta \leq \dim f(E) \leq \alpha$ whenever $E \subset \mathbb{H}^n$ with $\dim E = s$ and $f$ is $K$-qc.
3. For each $s \in [1, Q]$ there exists $E \subset \mathbb{H}^n$ compact such that $GQC \dim_{\mathbb{H}^n} E = \dim E = s$. 

Proof sketch: For $s \leq 2^n + 1$, choose $E$ to be the union of line segments along integral curves of the horizontal vector field $X_1$ emanating from a Cantor set in the $z_2 \cdots z_n$ subspace. Criteria of the Bourdon–Pansu theorem hold.

For $2^n + 1 \leq s \leq Q$, let $E = \bigcup_{r \in C} \partial B_{\mathbb{H}^n}(o, r)$ be a Cantor set's worth of Korányi spheres $\partial B_{\mathbb{H}^n}(o, r) = \{ p \in \mathbb{H}^n : d_{\mathbb{H}^n}(o, p) = r \}$. Apply a version of the Bourdon–Pansu criterion for such foliations.

4. (T, 2008) Kovalev's theorem holds for the Assouad dimension on $\mathbb{H}^n$.

In other words, if $E \subset \mathbb{H}^n$, $\dim A_E < 1$ then $GQC \dim_{\mathbb{H}^n} E = \dim E = s$. 

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3. For each $s \in [1, Q] \exists E \subset \mathbb{H}^n$ compact s.t. $GQC \dim_{\mathbb{H}^n} E = \dim E = s$.

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For $2n + 1 \leq s \leq Q$, let $E = \bigcup_{r \in C} \partial B_H(o, r)$ be a Cantor set’s worth of Korányi spheres $\partial B_H(o, r) = \{p \in \mathbb{H}^n : d_H(o, p) = r\}$. Apply a version of the Bourdon–Pansu criterion for such foliations.
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Quasiconformal flows

Recall the standard left invariant vector fields $X_1, Y_1, \ldots, X_n, Y_n$ generating $H\mathbb{H}^n$.

$$X_j(p) = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j(p) = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$$

Korányi and Reimann (1995) give conditions on a vector field $V$ which guarantee that the time $s$ map $p(0) \mapsto p(s)$ associated to the ODE $\dot{p} = V(p)$ is $K(s)$-qc. Basic structure is

$$V = \varphi\, T + \frac{1}{4} \sum_{j=1}^{n} (X_j\varphi\, Y_j - Y_j\varphi\, X_j) \quad \text{for } \varphi \in C^\infty(\mathbb{H}^n)$$

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with additional restrictions on the second horiz derivatives of $\varphi$.

**Examples:** $\varphi = 1$, $\varphi = x_k$, $\varphi = y_k$, $k = 1, \ldots, n$ generate left translation flows

$$\varphi = 2t \mapsto V = 2tT + \sum_j (x_j \ X_j + y_j \ Y_j)$$

generates dilation flow $p_0 \mapsto \delta_{es}(p_0)$
Fix a ball $B(p, R)$ and a family of disjoint balls $B_j = B(p_j, r_j) \subset B(p, r)$.

Choose potentials $\varphi_j$ generating contractive similarities with fixed points at the $p_j$, and define $\varphi = \sum_j \chi_j \varphi_j$ for suitable smooth cutoff functions $\chi_j$.

For suitable $s$, the time $s$ map $F$ associated to $\varphi$ is qc, fixes $\partial B(p, r)$, and contracts each $B_j$. Moreover, $F$ is conformal outside of $B$ and inside each $B_j$. Iterate.

As the flow method only works for smooth maps, use compactness of the space of normalized $K$-qc maps to pass to the limit.
Theorem (Pansu)

Every qc map $f$ of domains in $\mathbb{H}^n$ is differentiable almost everywhere in the following sense: for a.e. $p \in \Omega$ the limit

$$\delta_{1/r} \circ \ell_{f(p)}^{-1} \circ f \circ \ell_p \circ \delta_r$$

converges locally uniformly to a grading preserving automorphism $Df_p$ of $\mathbb{H}^n$. 

Intuition: The CC ball at $p$ of radius $0 < r \ll 1$ is comparable to the box $[-r, r]^2 \times [-r^2, r^2]$, sheared so that the $[-r, r]^2$ factor lies along the subspace $H_p$. Metric quasiconformality $\Rightarrow$ at a point of differentiability, the size $r$ and size $r^2$ directions cannot be interchanged by $f$. 

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converges locally uniformly to a grading preserving automorphism $Df_p$ of $\mathbb{H}^n$.

In particular, it follows that $f$ is a.e. diff’ble in horizontal directions, and that $f$ is a generalized contact map, $d_{hf_p} = (Df_p)_*|_{H_p\mathbb{H}^n}$ maps $H_p\mathbb{H}^n$ to $H_{f(p)}\mathbb{H}^n$ a.e.

Intuition: The CC ball at $p$ of radius $0 < r \ll 1$ is comparable to the box $[-r, r]^{2n} \times [-r^2, r^2]$, sheared so that the $[-r, r]^{2n}$ factor lies along the subspace $H_p\mathbb{H}^n$. Metric quasiconformality $\Rightarrow$ at a point of differentiability, the size $r$ and size $r^2$ directions cannot be interchanged by $f$. 

For $f = (f_1, \ldots, f_{2n+1})$ quasiconformal, we have

$$||d_h f||^Q \leq K \det Df \quad \text{a.e.}$$

In particular, each component $f_j$ lies in the local horiz Sobolev space $W_{loc}^{1,Q}(\Omega)$. 


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**Definition**

Given a metric space \( Y \), fix an isometric embedding \( \kappa \) of \( Y \) into a Banach space \( V \). Then

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W^{1,p}(\Omega : Y) := \{ f \in W^{1,p}(\Omega : V) : f(p) \in \kappa(Y) \text{ a.e.} \}
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For instance when \( Y = \mathbb{H}^n \) we can choose \( V = \ell^\infty \) (Fréchet embedding).

Each \( f \) qc lies in the metric space-valued Sobolev space \( W^{1,Q}_{loc}(\Omega : \mathbb{H}^n) \), in fact, \( \exists p(\mathbb{H}^n, K) > Q \) s.t. \( f \) \( K \)-qc in \( \mathbb{H}^n \) lies in \( W^{1,p}_{loc} \) for all \( p < p(\mathbb{H}^n, K) \).
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We’ll consider the exceptional sets problem for \( W^{1,p} \) maps, \( p > Q \), for generic elements in foliations of \( \mathbb{H}^n \) by left or right cosets of horizontal subgroups.
A smooth submanifold $\Sigma \subset \mathbb{H}^n$ of (topological) dimension $p \leq n$ has CC Hausdorff dimension $p$ if and only if $\Sigma$ is horizontal. Otherwise, it has CC dimension $p + 1$. On the other hand, every submanifold $\Sigma$ of topological dimension $> n$ satisfies $\dim \Sigma = p + 1$.

For example, a (smooth) curve $\gamma$ in $\mathbb{H}^1$ is horizontal if and only if $\dim \gamma = 1$, while any surface has CC dimension 3.
Let $V$ be a left invariant horiz vector field, $V = \{\exp(sV) : s \in \mathbb{R}\}$ the corresponding one-parameter horizontal subgroup.

The Euclidean complement $W$ of $V$ is a normal subgroup of $\mathbb{H}^n$, and $\mathbb{H}^n$ admits two semidirect decompositions: $\mathbb{H}^n = W \rtimes V$ or $\mathbb{H}^n = V \ltimes W$. These decompositions induce projection maps $p^L_W : \mathbb{H}^n \rightarrow W$ and $p^R_W : \mathbb{H}^n \rightarrow W$. 

Recall: $\pi : X \rightarrow W$ is locally s-regular if $\pi$ is locally Lipschitz, onto, and for each compact $K$ $\exists C > 0$ s.t. for every $B_{r} \subset W$, $\pi^{-1}(B_{r}) \cap K$ is covered by $\leq Cr^{s}$ balls of radius $Cr$. 


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**Left coset foliation** $\{a \ast V = (p^L_\mathbb{W})^{-1}(a) : a \in \mathbb{W}\}$: fibers are all 1-dimensional, $(\mathbb{W}, d_{cc})$ is $(2n + 1)$-dimensional, $p^L_\mathbb{W} : (\mathbb{H}^n, d_{cc}) \to (\mathbb{W}, d_{cc})$ is **not** Lipschitz.
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The Euclidean complement $W$ of $V$ is a normal subgroup of $H^n$, and $H^n$ admits two semidirect decompositions: $H^n = W \times V$ or $H^n = V \times W$. These decompositions induce projection maps $p_L^W : H^n \to W$ and $p_R^W : H^n \to W$.

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**Right coset foliation** $\{V \ast a = (p_R^W)^{-1}(a) : a \in W\}$: fibers are generically 2-dimensional, $W$ can be equipped with a quotient metric $d$ w.r.t. which $W$ is $(2n)$-dimensional, $p_R^W : (H^n, d_{cc}) \to (W, d)$ is Lipschitz.
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**Lemma:** $p^L_W : (H^n, d_{cc}) \to (W, d_E)$ and $p^R_W : (H^n, d_{cc}) \to (W, d)$ are both locally David–Semmes 2-regular.

Recall: $\pi : X \to W$ is *locally $s$-regular* if $\pi$ is locally Lipschitz, onto, and for each compact $K \ni C > 0$ s.t. for every $B_r \subset W$, $\pi^{-1}(B) \cap K$ is covered by $\leq Cr^{-s}$ balls of radius $Cr$. 

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From the metric space results of Lecture IV we obtain conclusions about generic dimension increase for images of cosets of horizontal subgroups. These results are stated for the quotient metric $d$ on $W$ in the right coset foliation $V ⋊ W$ and for the Euclidean metric on $W$ in the left coset foliation $W ⋉ V$.

**Corollary**

Let $Ω ⊂ H$ and let $f ∈ W^1_p(Ω : Y)$, $p > 4$, be continuous. For the horizontal subgroup $V = \{(x, 0, 0) : x ∈ R\}$ with complementary vertical subgroup $W = \{(0, y, t) : y, t ∈ R\}$, and for $2 < α ≤ 2p_p − 2$ we have

$$\dim d\{a ∈ W : \dim f(V ∗ a) ≥ α\} ≤ 2 − p(1 − 2^α),$$

$$\dim dE\{a ∈ W : \dim f(a ∗ V) ≥ α\} ≤ 2 − p(1 − 2^α).$$
Exceptional sets for $\mathbb{H}^n$ Sobolev dimension distortion

joint work with Z. Balogh and K. Wildrick

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Corollary

Let $\Omega \subset \mathbb{H}$ and let $f \in W^{1,p}(\Omega : Y)$, $p > 4$, be continuous. For the horizontal subgroup $V = \{(x, 0, 0) : x \in \mathbb{R}\}$ with complementary vertical subgroup $W = \{(0, y, t) : y, t \in \mathbb{R}\}$, and for $2 < \alpha \leq \frac{2p}{p-2}$ we have

$$
\dim_d \{a \in W : \dim f(V * a) \geq \alpha\} \leq 2 - p\left(1 - \frac{2}{\alpha}\right),
$$

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\dim_{d_E} \{a \in W : \dim f(a * V) \geq \alpha\} \leq 2 - p\left(1 - \frac{2}{\alpha}\right).
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The dimension estimate for left coset foliations on the previous slide is not optimal (since the David–Semmes regularity exponent of the foliation differs from the fiber dimension).
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**Theorem**

*Same assumptions as above. For* $1 \leq \alpha \leq \frac{p}{p-3}$ *we have*

$$\dim_{d_E} \{ a \in W : \dim f(a \ast V) \geq \alpha \} \leq \begin{cases} 2 - \frac{p}{2} \left(1 - \frac{1}{\alpha}\right) & 1 \leq \alpha < \frac{p}{p-2} \\ 3 - p \left(1 - \frac{1}{\alpha}\right) & \frac{p}{p-2} \leq \alpha \leq \frac{p}{p-3} \end{cases}$$
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**Example:** For $p > 4$ and $1 < \alpha < \frac{p}{p-2}$, $\exists F \subset W$ compact and $f \in W^{1,p}(\mathbb{H} : \mathbb{R}^2)$ continuous s.t. $\dim f(a \ast V) \geq \alpha \ \forall \ a \in F$ and $0 < \mathcal{H}^{2-p(1-1/\alpha)}_{dE}(F) < \infty$.  

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For left cosets of $m$-dimensional horizontal subgroups the result reads as follows:

**Theorem**

Let $\Omega \subset \mathbb{H}^n$ and let $f \in W^{1,p}(\Omega : Y)$, $p > Q$, be continuous. For $m \in \{1, \ldots, n\}$, an $m$-dimensional horizontal subgroup $\mathbb{V}$, and $m \leq \alpha \leq \frac{pm}{p-Q+m}$ we have

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Again, we used the Eucl metric on $\mathbb{W}$ to estimate the size of the exceptional set.
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Again, we used the Eucl metric on $\mathbb{W}$ to estimate the size of the exceptional set.

The estimate recovers the borderline value $Q - m - 1 = 2n + 1 - m = \dim_{d_E} \mathbb{W}$ when $\alpha \to m$, and reaches zero when $\alpha$ tends to the universal upper dimension estimate $\frac{pm}{p-Q+m}$.

The statement is only on the level of dimension, not measure. Also, there is a gap between the positive result and the $\mathbb{H}^1$ example which we are not able to resolve. Such a result would follow, for instance, from a positive answer to the following:
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The statement is only on the level of dimension, not measure. Also, there is a gap between the positive result and the \( \mathbb{H}^1 \) example which we are not able to resolve. Such a result would follow, for instance, from a positive answer to the following:

**Question:** Does there exist a locally David–Semmes 1-regular surjection from \( \mathbb{H}^n \)?
Thanks for your attention!