Distortion of dimension by Sobolev and quasiconformal mappings

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17 June 2014

I. Introduction and overview, quasiconformal maps of $\mathbb{R}^n$ and their effect on Hausdorff dimension
II. Global quasiconformal dimension in $\mathbb{R}^n$
III. Conformal dimension of metric spaces
IV. Sobolev dimension distortion in $\mathbb{R}^n$ and in metric spaces
V. QC and Sobolev dimension distortion in the sub-Riemannian Heisenberg group

joint with Z. Balogh and R. Monti / Z. Balogh and K. Wildrick

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Let $p > n$ s.t. $f \in W^{1,p}_{loc}$. Hölder's inequality $\Rightarrow$

$$\mathrm{Vol}\, f(Q_j) = \int_{Q_j} \det Df \le \int_{Q_j} ||Df||^n \le (\mathrm{Vol}\, Q_j)^{1-\frac{n}{p}} \left(\int_{Q_j} ||Df||^p \right)^{\frac{n}{p}}$$

Since f is quasisymmetric, $\exists\, C$ s.t. $\mathrm{diam}\, f(Q_j) \le C(\mathrm{Vol}\, f(Q_j))^{1/n}\ \forall\, j$. Since $\mathrm{Vol}\, Q_j \simeq (\mathrm{diam}\, Q_j)^n$, we conclude that

$$\mathrm{diam}\, f(Q_j) \le C(\mathrm{diam}\, Q_j)^{1-\frac{n}{p}} \left(\int_{Q_j} ||Df||^p \right)^{\frac{1}{p}} \quad \cdots$$

Quasisymmetry was only used to derive the highlighted estimate. In fact, the **Morrey–Sobolev embedding theorem** asserts that this estimate holds for arbitrary Sobolev mappings $f \in W^{1,p}(\Omega : \mathbb{R}^N)$, $\Omega \subset \mathbb{R}^n$.

From the discussion on the previous slide we conclude that the estimate

$$\dim f(E) \leq \frac{ps}{p-n+s} \qquad E \subset \Omega,\, \dim E = s$$

holds for arbitrary mappings $f \in W^{1,p}(\Omega : \mathbb{R}^N)$, $\Omega \subset \mathbb{R}^n$, $p > n$.

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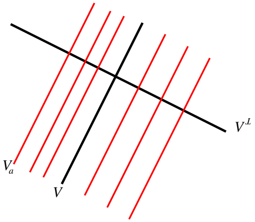
Recall that Sobolev functions admit **ACL** representatives. f is ACL if the restriction of f to almost every line V_a parallel to a fixed line V is (locally) absolutely continuous. In this case,

$$\dim f(V_a) \leq 1$$

for $\mathcal{L}^{n-1}$-a.e. $a \in V^{\perp}$.



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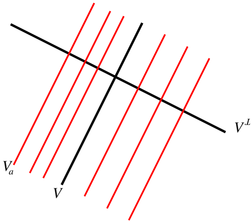

Problem

Can we interpolate between these two statements?

Let $f \in W^{1,p}(\Omega : \mathbb{R}^N)$, $p > n$, and let α be given, $m < \alpha \leq \frac{pm}{p-n+m}$.

Problem

For how many $a \in V^\perp$ can the dimension of $f(V_a)$ exceed α?

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It turns out that the set of such a is a $\mathcal{H}^\beta$-null set for a suitable $\beta \in [0, n-m]$. We expect that $\beta \to n-m$ as $\alpha \to m$ and $\beta \to 0$ as $\alpha \to \frac{pm}{p-n+m}$.

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Theorem A

Let $\beta = \beta(\alpha, p) = (n-m) - p\left(1 - \frac{m}{\alpha}\right)$. Then

$$\mathcal{H}^\beta(\{a \in V^\perp : \mathcal{H}^\alpha(f(V_a)) > 0\}) = 0$$

whenever $m < \alpha \leq \frac{pm}{p-n+m}$.

Applications to quasiconformal mappings

Recall that $p(n, K)$ denotes the supremum of all $p > n$ s.t. every K-qc map of $\mathbb{R}^n$ is in $W^{1,p}_{loc}$. Letting $p \to p(n, K)$ in the preceding theorem gives

Corollary

Let f be a K-qc map in $\mathbb{R}^n$. For each α, $m < \alpha \leq \frac{p(n,K)m}{p(n,K)-n+m}$

$$\dim\{a \in V^{\perp} : \dim f(V_a) > \alpha\} \leq (n-m) - p(n,K)\left(1 - \frac{m}{\alpha}\right) < m\left(\frac{n}{\alpha} - 1\right).$$

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The exact value of $p(n, K)$ is known only when $n = 2$. Astala (1994) showed that $p(2, K) = \frac{2K}{K-1}$. We obtain

Corollary

Let f be a K-qc map of planar domains, $V \subset \mathbb{R}^2$ a line. For $1 < \alpha \leq \frac{2K}{K+1}$,

$$\dim\{a \in V^{\perp} : \dim f(V_a) > \alpha\} \leq 1 - \tfrac{2K}{K-1}\left(1 - \frac{1}{\alpha}\right) = \frac{2K - (K+1)\alpha}{\alpha(K-1)}.$$

Theorem A estimates the size of the set of subspaces parallel to a fixed m-dimensional subspace, whose dimensions can be increased by a definite amount under a fixed Sobolev mapping. Our next result shows that the estimate is sharp, for all choices of the data.

Theorem B

For each $p > n$, $m \in \{1, \ldots, n-1\}$, and $\alpha \in (m, \frac{pm}{p-n+m}]$, there exists a compact set $F \subset V^{\perp}$ and a $W^{1,p}$ map $f : \mathbb{R}^n \to \mathbb{R}^N$ (for sufficiently large N) such that

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and $\dim f(V_a) \geq \alpha$ for all $a \in F$.

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The proof is nonconstructive. We build a family of $W^{1,p}$ mappings $f : \mathbb{R}^n \to \mathbb{R}^N$ by summing random vectors in the unit ball of $\mathbb{R}^N$ over the dyadic cubes in $\mathbb{R}^n$. We then prove that with positive probability one of these maps has the desired property. The idea is due to Kaufman, who proved sharpness of the universal statement $\dim f(E) \leq p \dim E/(p - n + \dim E)$ by this method.

The proof of Theorem A consists of carrying out the proof of the Gehring–Väisälä theorem ‘in parallel’, along many fibers of the projection $P_{V^\perp} : \mathbb{R}^n \to V^\perp$.

To do this effectively, we work with dyadic Hausdorff measures. This has two advantages:

- we can always consider essentially disjoint coverings, eliminating any need for covering theorems,
- the dyadic decomposition is well adapted to the orthogonal decomposition $\mathbb{R}^n = V \oplus V^\perp$: assuming that $V = \mathbb{R}^m \times \{0\}$ and $V^\perp = \{0\} \times \mathbb{R}^{n-m}$, the projection of a dyadic cube is dyadic and the fiber over a dyadic cube is partitioned into an essentially disjoint union of dyadic cubes.

The second key idea in the proof is to integrate the desired covering estimates across the exceptional set $E \subset V^{\perp}$ with respect to an appropriate measure. We choose such a measure using Frostman's lemma.

Lemma (Frostman)

Let E be a Borel set in $\mathbb{R}^n$ such that $\mathcal{H}^s(E) > 0$ for some $s > 0$. Then there exists a nontrivial Borel measure μ supported on E s.t. $\mu(B(x, r)) \leq r^s$ for all $x \in E$ and $r > 0$.

Proof of Thm A: $\mathcal{H}^\beta(E_\alpha) = 0$, $E_\alpha := \{a : \mathcal{H}^\alpha(f(V_a)) > 0\}$

First observe that we may WLOG restrict to a compact set $K \subset \mathbb{R}^n$.

Suppose $\mathcal{H}^\beta(E_\alpha) > 0$. Choose a Frostman measure μ on E_α ($\mu(B_{V^\perp}(a, r)) \leq r^\beta$).
Cover E_α with dyadic cubes $\{R_i\}$ in $V^\perp$ s.t. $\sum_i (\operatorname{diam} R_i)^\beta < \infty$.

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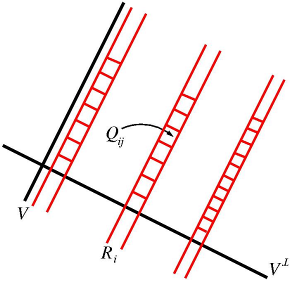
To each R_i, we can associate at most $N_i = C(\operatorname{diam} R_i)^{-m}$ dyadic cubes $\{Q_{ij}\}$ in $\mathbb{R}^n$ s.t. $P_{V^\perp}(Q_{ij}) = R_i$. Moreover, $\operatorname{diam} Q_{ij} \simeq \operatorname{diam} R_i$ and

$$Q_{ij} \subset U_\delta := \ \delta\text{-nbhd of } P_{V^\perp}^{-1}(E_\alpha) \cap K.$$



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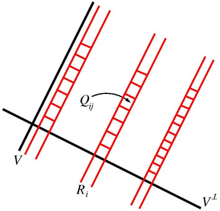

For each a, cover $f(V_a \cap K)$ with $\{f(Q_{ij}) : a \in R_i\}$:

$$\mathcal{H}^\alpha_\epsilon(f(V_a \cap K)) \leq \sum_{j : a \in R_i} (\operatorname{diam} f(Q_{ij}))^\alpha.$$

Now integrate over E_α w.r.t. μ. The proof is complete if we can show that

$$\int \mathcal{H}^\alpha_\epsilon(f(V_a \cap K))\, d\mu(a) \to 0 \qquad \text{as } \epsilon \to 0,$$

since then $\mathcal{H}^\alpha(f(V_a \cap K)) = 0$ for μ-a.e. $a \in E_\alpha$.

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We elaborate on the Gehring–Väisälä proof.

$$\int \mathcal{H}^\alpha_\epsilon(f(V_a \cap K))\, d\mu \le \sum_i \sum_{j=1}^{N_i} \mu(R_i)(\operatorname{diam} f(Q_{ij}))^\alpha$$

$$\le C \sum_i \sum_{j=1}^{N_i} (\operatorname{diam} R_i)^\beta (\operatorname{diam} Q_{ij})^{\alpha(1-\frac{n}{p})} \left(\int_{Q_{ij}} |\nabla f|^p \right)^{\frac{\alpha}{p}}$$

by Frostman's theorem and the Morrey–Sobolev inequality.

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by Frostman's theorem and the Morrey–Sobolev inequality. The estimate $N_i \le C(\mathrm{diam}\, R_i)^{-m}$ and the choice of β give

$$\int \mathcal{H}^\alpha_\epsilon(f(V_a \cap K))\, d\mu \le C \left(\sum_i (\mathrm{diam}\, R_i)^\beta \right)^{1-\frac{\alpha}{p}} \left(\int_{P^{-1}_{V^\perp}(\cup_i R_i) \cap K} |\nabla f|^p \right)^{\frac{\alpha}{p}}$$

where the first term is finite and the second tends to zero with δ.

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Let $f : X \to Y$ be a mapping between metric spaces. A Borel function $g : X \to [0, \infty]$ is an *upper gradient* of f if

$$d_Y(f(\gamma(a)), f(\gamma(b))) \le \int_\gamma g\, ds \qquad \forall\, \gamma : [a, b] \to X \text{ rectifiable.}$$

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Examples

(1) $g = |\nabla f|$ is an upper gradient for any C^1 (Lipschitz) f.

(2) $g = |\nabla_H f|$ is an u.g. for any $f : \mathbb{H}^n \to \mathbb{R}$ with cont's horiz partial deriv's.

(3) g an upper gradient of f and $h \ge g \Rightarrow h$ an upper gradient of f.

(4) set of upper gradients of a given function f preserved under convex combinations, pointwise minima, is localizable, . . .

Sobolev dimension distortion in metric measure spaces

Let (X, d, μ) be a metric measure space. The *(Newtonian–)Sobolev space* $N^{1,p}(X : Y)$ consists all p-integrable $f : X \to Y$ with p-integrable upper gradient.

The integrability of f will not be important: we will only be interested in cont's maps with L^p upper gradients. The only property of such maps which we will use is the Morrey–Sobolev inequality, which holds under suitable assumptions on X.

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Morrey–Sobolev: Assume X Q-regular, supports a Q-Poincaré inequality. For $f \in N^{1,p}(X : Y)$, $p > Q$, we have $\operatorname{diam} f(B) \le C(\operatorname{diam} B) \left(⨍_{\tau B} g^p \, d\mu \right)^{1/p}$ $\forall B \subset X$.

Definition

A metric measure space (X, d, μ) *supports a Q-Poincaré inequality* if $\exists\ C > 0$ and $\tau \ge 1$ s.t. $\forall$ f cont's w/ upper gradient g,

$$⨍_B |f - f_B| \, d\mu \le C(\operatorname{diam} B)(⨍_{\tau B} g^Q \, d\mu)^{1/Q} \qquad \forall B \subset X.$$

The same argument as in the Euclidean case gives universal upper bounds for dimension increase by Newtonian–Sobolev mappings.

Theorem C

Assume X is Ahlfors Q-regular and supports a Q-Poincaré inequality, $Q \geq 1$. If $f : X \to Y$ is continuous with an upper gradient in $L^p(X)$, $p > Q$, then

$$\dim f(E) \leq \frac{p \cdot \dim E}{p - Q + \dim E} \qquad E \subset X.$$

Can we obtain any analog in metric spaces for the generic dimension estimates for e.g., foliations of $\mathbb{R}^n$ by affine subspaces?

Generic dimension estimates in metric measure spaces

Definition (David–Semmes)

A mapping $\pi : X \to W$ between metric spaces is *locally s-regular* if π is a locally Lipschitz surjection and for each compact K $\exists$ $C > 0$ s.t. for every ball $B \subset W$, $\pi^{-1}(B) \cap K$ can be covered by at most Cr^{-s} balls in X of radius Cr.

Examples

(1) $P_{V^\perp} : \mathbb{R}^n \to V^\perp$ is locally $(\dim V)$-regular.

(2) Riemannian submersions

(3) Projection maps onto cosets of horizontal or vertical subgroups of the Heisenberg group $\mathbb{H}^n$

Generic dimension estimates in metric measure spaces

Theorem D

Same assumptions as in Theorem C. Assume $\pi : X \to W$ is locally David–Semmes s-regular for some $0 < s < Q$. Then, for each α with $s < \alpha < ps/(p - Q + s)$,

$$\dim\{a \in W : \dim f(\pi^{-1}(a)) \geq \alpha\} \leq (Q - s) - p(1 - \tfrac{s}{\alpha}).$$

Remarks

(1) Theorem D reduces to Theorem A when $X = \mathbb{R}^n$, $W = V^\perp$, $\pi = P_{V^\perp}$. In this case, $s = m = \dim V = \dim V_a$ for all $a \in V^\perp$.

(2) In the case $W = \mathbb{R}^n$ we can relax the assumptions on π; it suffices to assume that the fibers $\pi^{-1}(a)$ are s-regular (even a bit less).

(3) In some settings, the result is necessarily not sharp. This occurs when the fibers $\pi^{-1}(a)$ have dimension strictly less than s.

Foliations of $\mathbb{H}^n$

Let V be a left invariant horizontal vector field on $\mathbb{H}^n$, $\mathbb{V} = \{\exp(sV) : s \in \mathbb{R}\}$ the corresponding one-parameter horizontal subgroup.

The Euclidean complement $\mathbb{W}$ of $\mathbb{V}$ is a normal subgroup of $\mathbb{H}^n$, and $\mathbb{H}^n$ admits two semidirect decompositions: $\mathbb{H}^n = \mathbb{W} \ltimes \mathbb{V}$ or $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$.

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These decompositions induce 'projection' mappings $p^L_{\mathbb{W}} : \mathbb{H}^n \to \mathbb{W}$ and $p^R_{\mathbb{W}} : \mathbb{H}^n \to \mathbb{W}$. We consider the *left* and *right coset foliations* $\{a * \mathbb{V} = (p^L_{\mathbb{W}})^{-1}(a)\}$ and $\{\mathbb{V} * a = (p^R_{\mathbb{W}})^{-1}(a)\}$ for $a \in \mathbb{W}$.

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Left coset foliation: fibers are all 1-dimensional, $(\mathbb{W}, d_{cc})$ is $(2n+1)$-dimensional, $p^L_{\mathbb{W}} : (\mathbb{H}^n, d_{cc}) \to (\mathbb{W}, d_{cc})$ is **not** Lipschitz.

Right coset foliation: fibers are generically 2-dimensional, $\mathbb{W}$ can be equipped with a quotient metric d w.r.t. which $\mathbb{W}$ is $(2n)$-dimensional, $p^R_{\mathbb{W}} : (\mathbb{H}^n, d_{cc}) \to (\mathbb{W}, d)$ is Lipschitz.

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Lemma: $p^L_{\mathbb{W}} : (\mathbb{H}^n, d_{cc}) \to (\mathbb{W}, d_E)$ and $p^R_{\mathbb{W}} : (\mathbb{H}^n, d_{cc}) \to (\mathbb{W}, d)$ are both locally David–Semmes 2-regular.