Distortion of dimension by Sobolev and quasiconformal mappings

J. Tyson

17 June 2014

I. Introduction and overview, quasiconformal maps of $\mathbb{R}^n$ and their effect on Hausdorff dimension
II. Global quasiconformal dimension in $\mathbb{R}^n$
III. Conformal dimension of metric spaces
IV. Sobolev dimension distortion in $\mathbb{R}^n$ and in metric spaces
V. QC and Sobolev dimension distortion in the sub-Riemannian Heisenberg group

In this lecture we discuss Pansu's *conformal dimension*.

Definition (Pansu, 1989)

Let X be a metric space. The *conformal dimension* of X is

$$C\dim X = \inf\{\dim Y \ : \ Y \text{ a metric space, } Y \overset{qs}{\sim} X\}.$$

Recall: $f : X \to Y$ is η*-quasisymmetric (qs)* if

$$|f(x) - f(a)| \leq \eta(t)|f(x) - f(b)|$$

whenever $x, a, b \in X$ satisfy $|x - a| \leq t|x - b|$ and $t > 0$.

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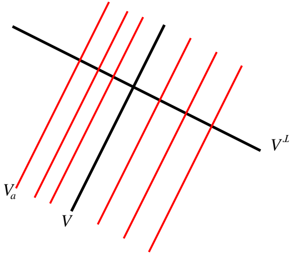
From last time: $C\dim X = 0$ if $X \subset \mathbb{R}^n$ is a self-similar Cantor set or has dim < 1.

Lower bounds for conformal dimension

The general philosophy is: *lower bounds on $C\dim X$ arise from "well distributed" families of curves inside X.*

The model case is the foliation of $\mathbb{R}^n$ by lines parallel to a fixed direction.



Proposition A (after M. Bourdon, P. Pansu)

Let (X, d, μ) be a doubling metric measure space and let $1 < p < \infty$. Let Γ be a family of curves in X equipped with a probability measure ν s.t.

(i) *the support of Γ is bounded,*

(ii) *the elements of Γ have diameters uniformly bounded away from zero, and*

(iii) *$\exists\, C > 0$ s.t. $\nu\{\gamma \in \Gamma : \gamma \cap B(x,r) \neq \emptyset\} \leq C\mu(B(x,r))^{1/p}$ $\forall\, B(x,r)$.*

Then $C\dim X \geq p'$, where $p' = \frac{p}{p-1}$.

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Then $C\dim X \geq p'$, *where* $p' = \frac{p}{p-1}$.

Remark

If μ is assumed s-regular, then (iii) can be replaced by

(iii') there exists $C > 0$ s.t. $\nu\{\gamma \in \Gamma : \gamma \cap B(x, r) \neq \emptyset\} \leq Cr^{s/p}\ \forall\, B(x, r)$.

If $p = \frac{s}{s-1}$ then the conclusion is $C\dim X = \dim X = s$. In this case $\frac{s}{p} = s - 1$.

If we can foliate a piece of X (of positive measure) by a family of curves which is "uniformly 1-codimensional", then X is minimal for conformal dimension.

Examples

1. $X = \mathbb{R}^n$ $(s = n)$, Γ foliation by parallel lines $V_a = V + a$, ν Lebesgue measure on $V^\perp$, $p = \frac{n}{n-1}$. Conclusion: $C\dim \mathbb{R}^n = \dim \mathbb{R}^n = n$.

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2. $X = \mathbb{H}^n$ Heisenberg group with left invariant Carnot–Carathéodory metric d_{cc} ($s = 2n + 2$), Γ foliation by integral curves of a horiz vector field V, Γ can be equipped with a measure ν s.t. the previous condition holds with $p = \frac{s}{s-1}$.

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$V \in \mathrm{span}\{X_1, Y_1, \ldots, X_n, Y_n\}$

$\gamma \in \Gamma$ satisfies $\gamma'(s) = V(\gamma(s))$ for all s

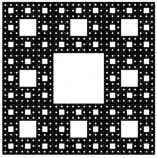
ν satisfies $|A| = \int_\Gamma \mathrm{length}(\gamma \cap A)\, d\nu(\gamma)$ for all $A \subset \mathbb{H}^n$

Conclusion: $C\dim(\mathbb{H}^n, d_{cc}) = \dim(\mathbb{H}^n, d_{cc}) = s$.

(due to Pansu)

Examples

3. X = Sierpiński carpet $SC \subset \mathbb{R}^2$



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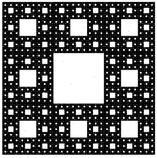

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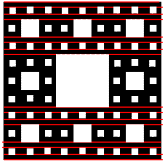

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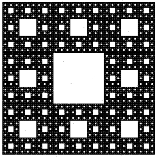

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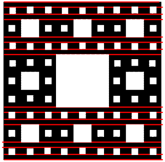
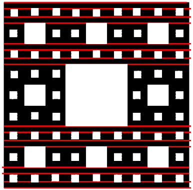


$\nu = \mathcal{H}^{\log 2/\log 3} \llcorner C$, $p = \frac{\log 8/\log 3}{\log 2/\log 3} = 3$. Conclusion: $C\dim SC \geq p' = \frac{3}{2} > 1$.

The estimate can be improved.



Consider $X_1 = C \times [0,1] \subset SC$. The measure

$$\mu_1 = \mathcal{H}^t \times \mathcal{L}^1 \llcorner X_1 \simeq \mathcal{H}^{t+1} \llcorner X_1 \qquad t = \frac{\log 2}{\log 3}$$

is Ahlfors regular on X_1, and

$$\nu(\{\gamma \in \Gamma : \gamma \cap B(x,r) \neq \emptyset\}) \leq C \mu_1(B(x,r))^{1/p}$$

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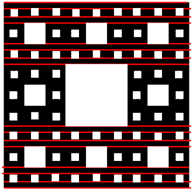

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with $p = \frac{1+t}{t}$. Hence $C\dim X_1 \geq p' = 1 + t$ and
$C\dim SC \geq C\dim X_1 = 1 + \frac{\log 2}{\log 3} > \frac{3}{2}$.

Lower bounds for conformal dimension can also be obtained using *moduli* of curve families.

Definition

Let Γ be a family of curves in a metric measure space (X, d, μ) and let $p \geq 1$. The *p-modulus of* Γ is

$$\mathrm{Mod}_p(\Gamma) = \inf \int_X \rho^p \, d\mu$$

where the infimum is taken over all *admissible* Borel functions $\rho : X \to [0, \infty]$, i.e., $\int_\gamma \rho \, ds \geq 1$ for all locally rectifiable $\gamma \in \Gamma$.

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Example

Let (Z, d, ν) be any compact mms and $X = Z \times [0, h]$ with the usual product metric and measure $\mu = \nu \otimes \mathcal{L}^1$. Let $\Gamma = \{\gamma_z : z \in Z\}$, $\gamma_z : [0, h] \to X$, $\gamma_z(s) = (z, s)$.

$$\mathrm{Mod}_p(\Gamma) = \frac{\nu(Y)}{h^{p-1}}.$$

"$\leq$": $\rho(z, s) = \frac{1}{h}$ is admissible
"$\geq$": apply Fubini's theorem and Hölder's inequality

Proposition B

Let (X, d, μ) *be a doubling metric measure space satisfying the upper mass bound* $\mu(B(x, r)) \leq r^s$ *for all* $x \in X$ *and* $r > 0$*. Assume that there exists a curve family* Γ *in* X *s.t.* $\mathrm{Mod}_p(\Gamma) > 0$ *for some* $1 < p \leq s$*. Then* $C\dim X \geq s$*.*

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Corollary

Assume that X is Ahlfors s-regular and supports a curve family Γ in X s.t. $\mathrm{Mod}_s(\Gamma) > 0$. Then $C\dim X = \dim X = s$.

Minimal sets for global qc dimension

For any $t \in (0, n-1)$ choose a compact t-regular set $Z \subset \mathbb{R}^{n-1}$ (e.g., Z a suitable self-similar Cantor set).

Let $X = Z \times [0,1] \subset \mathbb{R}^n$ equipped with product metric and measure $\mu = \mathcal{H}^t \times \mathcal{L}^1 \simeq \mathcal{H}^s$, $s = t + 1$.

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Then X is s-regular and supports a curve family Γ with $\mathrm{Mod}_p(\Gamma) > 0$ for any p.

Alternatively, the criteria of Proposition A hold with $\nu = \mathcal{H}^t \llcorner Z$ and $p = \frac{s}{s-1}$.

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Hence $C\dim X = \dim X = s$. In particular, $GQC\dim_{\mathbb{R}^n} X = s$.

Proposition A

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$$\nu\{\gamma \in \Gamma : \gamma \cap B(x,r) \neq \emptyset\} \leq C\mu(B(x,r))^{1/p} \quad \forall\, B(x,r).$$

Then $C\dim X \geq p'$.

Suppose $f : X \to Y$ is η-qs and $\dim Y < p'$. Uniform continuity of f implies all elements of $f(\Gamma)$ have diameters $\geq c' > 0$.

Cover Y with balls $\{B_i'\}$ s.t. $\{\frac{1}{5}B_i'\}$ are disjoint and $\sum_i (\operatorname{diam} B_i')^{p'} < \epsilon$.

Preimages of the balls B_i' under f are roughly balls; choose $B_i \subset X$ s.t. $B_i \subset f^{-1}(\frac{1}{5}B_i') \subset f^{-1}(B_i') \subset HB_i$ where $H = \eta(5)$. Note: $\{B_i\}$ are disjoint.

For each γ,

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Hölder's inequality and the doubling property of μ give

$$0 < C \leq \left(\sum_i (\operatorname{diam} B_i')^{p'}\right)^{1/p'} \left(\sum_i \mu(B_i)\right)^{1/p} \leq C\epsilon^{1/p'}\,\mu(A)^{1/p}$$

where A is a suitable neighborhood of the support of Γ. This leads to a contradiction if ϵ is sufficiently small.

Proposition B

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The proof is similar. For simplicity let $p = s$. Where previously we integrated over Γ w.r.t. ν, we now construct an admissible density ρ for the p-modulus of Γ. Choosing $\{B_i'\}$ and $\{B_i\}$ as before, set

$$\rho(x) = C \sum_i \frac{\operatorname{diam} B_i'}{r_i} \chi_{2HB_i}(x)$$

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for a suitable (large) constant C. Admissibility of ρ follows from the uniform lower bound on $\operatorname{diam} f(\gamma)$. Then

$$0 < \mathrm{Mod}_s(\Gamma) \le C \int_X \left(\sum_i \frac{\operatorname{diam} B_i'}{r_i} \chi_{2HB_i} \right)^s d\mu \le C \sum_i (\operatorname{diam} B_i')^s \frac{\mu(B_i)}{r_i^s} \to 0.$$

Annular linear connectivity and lower bounds for $C\dim$

Definition

(X, d) is *annularly linearly connected* (ALC) if $\exists\, L \geq 2$ s.t. any two points $x, y \in B(x_0, 2r) \setminus B(x_0, r/2)$ can be joined by an arc in $B(x_0, Lr) \setminus B(x_0, r/L)$.

Intuitively, X is annularly linearly connected if points in any annulus can be joined without going too far away from or too close to the center of the annulus. For instance, $\mathbb{R}^n$ is ALC if $n \geq 2$. Heisenberg group $(\mathbb{H}^n, d_{cc})$ is ALC for any n. Sierpiński carpet SC is ALC.

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Theorem (Mackay)

Let (X, d) be complete, doubling and annularly linearly connected. Then $C\dim X \geq 1 + \epsilon > 1$. The constant ϵ can be chosen to depend only on the doubling and annular linear connectivity constants of X.

X complete, doubling, ALC $\Rightarrow$ $C\dim X \geq 1 + \epsilon > 1$

Sketch of the proof: Start with a single arc γ in X. Choose an annulus A centered on γ. Pick two points x, y on $\gamma \cap A$ on opposite sides of the center.

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Repeat with both the old and the new arcs to get 4 arcs, 8 arcs, etc. Completeness $\Rightarrow$ $\exists$ a Cantor set's worth of arcs.

Hausdorff distance on this family of arcs Γ is bi-Lipschitz equivalent to a standard self-similar metric on the Cantor set C s.t. dimension $= t > 0$.

The standard self-similar measure on C pushes forward to a measure ν on Γ s.t.

$$\nu(\{\gamma \in \Gamma : \gamma \cap B(x,r) \neq \emptyset\}) \leq Cr^t.$$

By Proposition A, $C\dim X \geq 1 + c(t) > 1$. All steps are quantitative.

Additional remarks

1. As for global quasiconformal dimension, conformal dimension takes on no values strictly between zero and one.

In fact this follows from Kovalev's theorem, since nonseparable metric spaces have infinite conformal dimension and every separable metric space embeds isometrically into a real separable Banach space.

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2. We saw that under mild extra assumptions, the existence of a curve family of positive modulus suffices for minimality for conformal dimension. It is a much deeper fact that this criterion is also necessary.

Theorem (Keith–Laakso)

Let (X, d, μ) *be a compact metric measure space, Ahlfors s-regular for some* $s > 1$*. If* $C\dim_A X = s$ *then there exists some weak tangent* X_∞ *of* X *which carries a family* Γ *of curves with diameters* $\geq c > 0$ *and s.t.* $\mathrm{Mod}_s(\Gamma) > 0$*.*

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Quasisymmetric maps pass to weak tangents $\Rightarrow C\dim_A X_\infty \leq C\dim_A X$.

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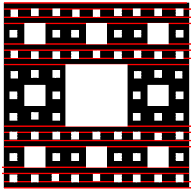
Conclusion: If X is s-regular for some $s > 1$ and has a weak tangent space X_∞ which contains a curve family of positive p-modulus for some $1 < p \leq s$, then $C\dim_A X = s$.

$$s = C\dim_A X_\infty \leq C\dim_A X \leq \dim_A X = s$$

3. Conformal dimension of the Sierpiński carpet



SC contains no curve families of positive p-modulus (for any p). Since SC is self-similar, neither does any weak tangent of SC.

By Keith–Laakso, it follows that $C\dim SC \le C\dim_A SC < \dim SC = \frac{\log 8}{\log 3}$.

$$1 < 1 + \frac{\log 2}{\log 3} \le C\dim SC < \dim SC = \frac{\log 8}{\log 3}$$

Question: $C\dim SC = ?$