Distortion of dimension by Sobolev and quasiconformal mappings

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I. Introduction and overview, quasiconformal maps of $\mathbb{R}^n$ and their effect on Hausdorff dimension
II. Global quasiconformal dimension in $\mathbb{R}^n$
III. Conformal dimension of metric spaces
IV. Sobolev dimension distortion in $\mathbb{R}^n$ and in metric spaces
V. QC and Sobolev dimension distortion in the sub-Riemannian Heisenberg group

In Lecture I, we introduced quasiconformal maps of $\mathbb{R}^n$ and described the effect of a K-qc map f of $\mathbb{R}^n$ on the Hausdorff dimension of a set $E \subset \mathbb{R}^n$.

We now consider the following problem.

Fix $E \subset \mathbb{R}^n$. What can be said about the values of $\dim f(E)$ *as f ranges over* **all** *quasiconformal mappings of $\mathbb{R}^n$?*

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Definition

The *global quasiconformal dimension* of a set $E \subset \mathbb{R}^n$ is

$$GQC\dim E = GQC\dim_{\mathbb{R}^n} E = \inf\{\dim f(E) \, : \, f \text{ a qc map of } \mathbb{R}^n\}$$

For instance, as discussed at the end of the last lecture, we have

$$GQC\dim_{\mathbb{R}^2} SG = 1.$$

Why the infimum?

Theorem (Bishop, 1999)

Let $E \subset \mathbb{R}^n$ with $\dim E > 0$ and let $\epsilon > 0$. Then there exists a quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ (if $n = 1$, quasisymmetric) so that $\dim f(E) > n - \epsilon$.

Since qc maps of $\mathbb{R}^n$ preserve sets of dimension zero, and in view of Bishop's theorem, the problem to determine

$$\sup\{\dim f(E) \,:\, f \text{ a qc map of } \mathbb{R}^n\} \qquad E \subset \mathbb{R}^n$$

is uninteresting.

Obvious (a priori) estimates:

$$\dim_T E \leq GQC\dim E \leq \dim E$$

where $\dim_T$ denotes the topological dimension.

Theorem (T, 1999)

For each $s \in [1, n]$ there exist compact sets $E \subset \mathbb{R}^n$ such that

$$GQC\dim E = \dim E = s.$$

We will prove this theorem in Lecture III, as a corollary of a more general statement about metric spaces *minimal for conformal dimension*.

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What about the case $0 < s < 1$? In the same paper where the above theorem appeared, we conjectured that the corresponding result for dimensions less than one was false. More precisely, we proposed the following

Conjecture

If $\dim E < 1$ *then* $GQC\dim E = 0$.

This conjecture was eventually proved by L. Kovalev. Today we discuss Kovalev's theorem and an earlier result establishing the conjecture for the *Assouad dimension*.

Assouad dimension

Definition

(X, d) is *doubling* if there exists a constant C so that every ball $B(x, r)$ contains no more than C points whose mutual distance is at least $r/2$.

By iteration, it follows that every $B(x, r)$ contains $\leq C'\epsilon^{-s}$ points whose mutual distance is at least ϵr. Here C' and $s > 0$ depend only on C (e.g., $s = \log_2 C$ suffices).

Definition

The *Assouad dimension* of (X, d) is

$$\dim_A X = \inf \left\{ s > 0 : \begin{array}{c} \exists\, C > 0 \text{ s.t. every } B(x,r) \text{ contains} \leq C\epsilon^{-s} \\ \text{points } x_1, \ldots, x_k \text{ s.t. } d(x_i, x_j) \geq \epsilon r \text{ if } i \neq j \end{array} \right\}.$$

Thus $\dim_A X < \infty$ iff X is doubling.

Assouad dimension: basic properties

Assouad dimension

- is a bi-Lipschitz invariant ($X \overset{BL}{\sim} Y \Rightarrow \dim_A X = \dim_A Y$),
- is monotone ($X \subset Y \Rightarrow \dim_A X \leq \dim_A Y$),
- is finitely stable ($\dim_A(X_1 \cup X_2) = \max\{\dim_A X_1, \dim_A X_2\}$),
- is equal to s for an Ahlfors s-regular space (X, d).

(X, d) is *Ahlfors s-regular* if it supports a Borel measure μ s.t. $\mu(B(x, r)) \simeq r^s$ for all $x \in X$ and $0 < r < \operatorname{diam} X$.

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Also $\dim X \le \dim_A X$ for any X and $\dim X = \dim_A X = s$ if X is s-regular. However,

- Lipschitz maps can increase Assouad dimension,
- $\dim_A X = \dim_A \overline{X}$ for any incomplete metric space X, where $\overline{X}$ denotes the metric completion of X, and hence
- $\dim_A$ is not countably stable, e.g., $\dim_A \mathbb{Q} = 1$.

In Lecture III we will prove the existence of compact subsets of $\mathbb{R}^n$ of any given dimension $s \in [1, n]$ which are minimal for global quasiconformal dimension. These sets will in fact be Ahlfors s-regular, hence minimal for both global quasiconformal Hausdorff and Assouad dimension.

The following theorem resolves the conjecture positively for Assouad dimension.

Theorem (T, 2001)

For any $E \subset \mathbb{R}^n$ such that $\dim_A E < 1$, $GQC\dim_A E = 0$.

In other words, if $E \subset \mathbb{R}^n$ has Assouad dimension < 1, then the Assouad dimension can be made arbitrarily small by global quasiconformal maps of $\mathbb{R}^n$.

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Remark

A set $E \subset \mathbb{R}^n$ is *porous* if there exists $c > 0$ so that every ball $B(x, r)$ contains a ball $B(z, cr)$ which is disjoint from E. It is not hard to see that E is porous if and only if $\dim_A E < n$.
Thus when $n = 1$ the theorem states that each porous subset of $\mathbb{R}$ has global quasiconformal Assouad dimension zero.

When $n \geq 2$ porosity is much weaker than having Assouad dimension < 1. We introduce another notion (*sparsity*) to record geometric information encoded in fixed upper bounds for $\dim_A E$. This notion allows us greater control on the location and size of sets contained in the complement of E.

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For a fixed integer $b \geq 2$ consider the tilings of $\mathbb{R}^n$ by b-adic cubes $Q = \{(y_1, \ldots, y_n) : b^{-m} j_i \leq y_i \leq b^{-m}(j_i + 1)\}$, $m, j_1, \ldots, j_n \in \mathbb{Z}$. Each such cube of side length b^{-m} has a well-defined *parent* of side length b^{1-m}. For a fixed integer k, $0 \leq k < b^n$, we say that E is (b,k)-*sparse* if at most k of the children of any b-adic cube intersect E.

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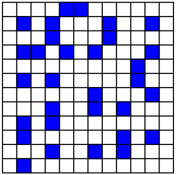
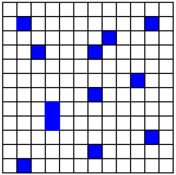
Proposition

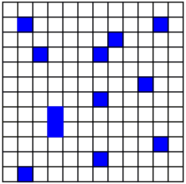
(1) $E \subset \mathbb{R}^n$ is porous iff E is $(b, b^n - 1)$-sparse for some sufficiently large b.
(2) For any given s, $0 < s \leq n$, $\dim_A E < s$ iff E is (b, k)-sparse for some sufficiently large b and for some $k < b^s$.





$\dim_A E < 1 \;\Rightarrow\; GQC\dim_A E = 0$: Sketch of the proof

Given $E \subset \mathbb{R}^n$, $\dim_A E < s < 1$. Choose $b \gg 1$, $k < b^s \ll b$ s.t. E (b,k)-sparse.



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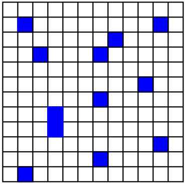

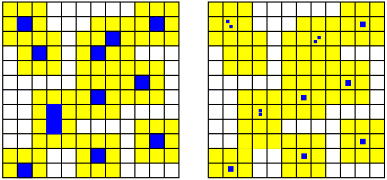

Since $k \ll b$ we can divide the "good" cubes into blocks B_j (blue), separated by large annuli A_j (white). In particular, 1-cube nbhds $\widehat{B}_j$ (yellow) of B_j are disjoint.

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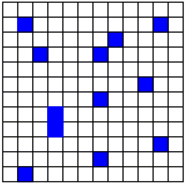

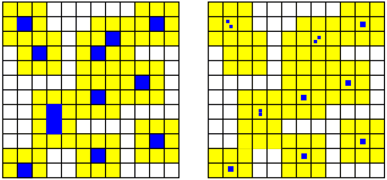

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Choose a qc map f_1 s.t. (i) $f_1 =$ identity on the boundary of $\widehat{B}_j$ and (ii) f_1 is a conformal scaling inside B_j into one of its constituent subblocks.

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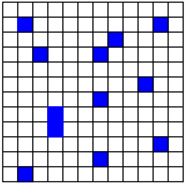

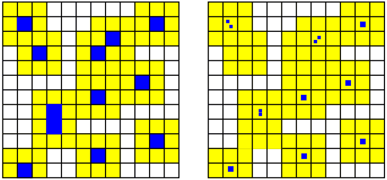

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Repeat inside each cube at the next level and iterate to get a sequence (f_m) of K-qc maps. Limit f_∞ is K-qc and maps E to a set E' of dimension $\le (1-\delta)s$ for some fixed $\delta > 0$. Assuming s close to $\dim_A E$ we get $\dim_A E' \le (1 - \frac{1}{2}\delta)\dim_A E$. Repeat as often as needed until dimension is $< \epsilon$.

The preceding proof works for the Assouad dimension since upper bounds on $\dim_A$ translate to uniform, scale-invariant information on the size of omitted sets at all locations and scales. As such, however, the argument breaks down completely for the Hausdorff dimension.

Using completely different methods coming from convex geometry, Kovalev verified the original conjecture.

Theorem (Kovalev, 2006)

Let $E \subset \mathbb{R}^n$ satisfy $\dim E < 1$. Then $GQC\dim E = 0$.

In other words, there exist qc maps f of $\mathbb{R}^n$ so that $\dim f(E)$ is arbitrarily small.

Kovalev's proof works if $\mathbb{R}^n$ is replaced by any real separable Banach space V, for quasisymmetric homeomorphisms $f : V \to V$.

L. Kovalev, 'Conformal dimension does not assume values between zero and one', *Duke Math. J.*, 2006

Theorem (Kovalev, 2006)

Let $E \subset \mathbb{R}^n$ *satisfy* $\dim E < 1$. *For each* $\epsilon > 0$ *there exists a qc map* $f : \mathbb{R}^n \to \mathbb{R}^n$ *s.t.* $\dim f(E) < \epsilon$.

The following proof is a simplified version of the original argument, incorporating ideas from the paper

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The qc map in question is obtained as the gradient of a convex function. Recall from Lecture I that a convex function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be *K-quasiuniformly convex* if $\nabla u : \mathbb{R}^n \to \mathbb{R}^n$ is K-qc. Alternatively, u is K-q.u. convex if $u \in W^{2,n}_{loc}$, u is not affine, and $||D^2 u||^n \le K \det D^2 u$ a.e.

The key observation is that q.u. convex functions form a **convex** family, in particular, the convolution of a K-q.u. convex function with a Radon measure is again a K-q.u. convex function. This fact is an easy consequence of Minkowski's determinantal inequality applied to the determinant of the Hessian.

Quasiconformal gradients and quasiuniform convexity

Example

For each $0 < a < 1$, the function $u(x) = \frac{|x|^{1+a}}{1+a}$ is a^{-1}-q.u. convex. Hence $f(x) = \nabla u(x) = |x|^{a-1}x$ is a^{-1}-qc.

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For ν Radon define

$$w^\nu(x) = \tfrac{1}{2}|x|^2 + (1+a)^{-1} \int (|x-y|^{1+a} - |y|^{1+a})\, d\nu(y).$$

Then w^ν is q.u. convex, and

$$T^\nu(x) := \nabla w^\nu(x) = x + \int |x-y|^{a-1}(x-y)\, d\nu(y)$$

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Proposition (Kovalev–Maldonado)

Let $\mathcal{M}^2$ denote the space of Radon measures on $\mathbb{R}^n$ with finite second moment. For each $\mu \in \mathcal{M}^2$ there exists $\nu \in \mathcal{M}^2$ s.t. $(T^\nu)_\# \nu = \mu$.

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Relabel these balls $B_{ki} = B(x_{ki}, r_{ki})$, $i \geq 1$, by replacing the two indices m, j with a single index i. Then $\{B_{ki} : i \geq 1\}$ covers E and $\sum_i r_{ki}^s (1 + |x_{ki}|^2) < 2^{-k}$.

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The map $T := T^\nu$, hence also T^{-1}, is qc. We claim that $\dim T^{-1}(E) < \epsilon$.

Set $C_{ki} = T^{-1}(B_{ki})$ and $d_{ki} = \operatorname{diam} C_{ki}$. Then

(1) $$\nu(C_{ki}) = \mu(B_{ki}) \geq r_{ki}^s \qquad \forall\, k, i.$$

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$$\nu(C_{ki}) = \mu(B_{ki}) \geq r_{ki}^s \qquad \forall\, k, i. \tag{1}$$

Fix $w_{ki}, z_{ki} \in C_{ki}$ with $|w_{ki} - z_{ki}| = d_{ki}$. Then

$$|T(w_{ki}) - T(z_{ki})| \leq \operatorname{diam} B_{ki} = 2r_{ki} \tag{2}$$

and

$$\begin{aligned}
|T(w_{ki}) - T(z_{ki})| &\geq \langle T(w_{ki}) - T(z_{ki}), \frac{w_{ki} - z_{ki}}{d_{ki}} \rangle \\
&\geq \int_{B(w_{ki}, d_{ki})} \left\langle (w_{ki} - y)|w_{ki} - y|^{a-1} - (z_{ki} - y)|z_{ki} - y|^{a-1}, \frac{w_{ki} - z_{ki}}{d_{ki}} \right\rangle \, d\nu(y) \\
&\geq \frac{1}{2} d_{ki}^a \nu(B(w_{ki}, d_{ki})).
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\end{aligned}$$

Since also $C_{ki} \subset B(w_{ki}, d_{ki})$ we conclude from (1) and (2) that

$$2r_{ki} \geq \frac{1}{2} d_{ki}^a \nu(C_{ki}) \geq \frac{1}{2} d_{ki}^a\, r_{ki}^s.$$

By the choice of a this is equivalent to $d_{ki}^\epsilon \leq C r_{ki}^s$.

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$$(2) \qquad |T(w_{ki}) - T(z_{ki})| \leq \text{diam}\, B_{ki} = 2r_{ki}$$

and

$$\begin{aligned}
|T(w_{ki}) - T(z_{ki})| &\geq \langle T(w_{ki}) - T(z_{ki}), \frac{w_{ki} - z_{ki}}{d_{ki}} \rangle \\
&\geq \int_{B(w_{ki}, d_{ki})} \left\langle (w_{ki} - y)|w_{ki} - y|^{a-1} - (z_{ki} - y)|z_{ki} - y|^{a-1}, \frac{w_{ki} - z_{ki}}{d_{ki}} \right\rangle \, d\nu(y) \\
&\geq \frac{1}{2} d_{ki}^a \nu(B(w_{ki}, d_{ki})).
\end{aligned}$$

Since also $C_{ki} \subset B(w_{ki}, d_{ki})$ we conclude from (1) and (2) that

$$2r_{ki} \geq \frac{1}{2} d_{ki}^a \nu(C_{ki}) \geq \frac{1}{2} d_{ki}^a \, r_{ki}^s.$$

By the choice of a this is equivalent to $d_{ki}^\epsilon \leq C r_{ki}^s$. Recall that $d_{ki} = \text{diam}\, C_{ki}$. Since $T^{-1}(E) \subset \cup_i C_{ki}$ for each k and $\sum_i d_{ki}^\epsilon \leq C \sum_i r_{ki}^s \to 0$ as $k \to \infty$, we conclude that $\mathcal{H}^\epsilon(T^{-1}(E)) = 0$ and so $\dim T^{-1}(E) < \epsilon$.

Proposition (Kovalev–Maldonado)

Let $\mathcal{M}^2$ denote the space of Radon measures on $\mathbb{R}^n$ with finite second moment. For each $\mu \in \mathcal{M}^2$ there exists $\nu \in \mathcal{M}^2$ s.t. $(T^\nu)_\# \nu = \mu$.

Sketch of the proof:

1. $\nu \mapsto (T^\nu)_\# \nu$ is a bijection on the space of finite atomic measures.

2. Continuity estimate:

$$\int (1 + |x|^2) \, d(T^\nu)_\# \nu(x) \geq \int (1 + |x|^2) \, d\nu(x)$$

for finite atomic ν.

3. Weak* convergence.

Theorem (Bishop, 1999)

Let $E \subset \mathbb{R}^n$ *with* $\dim E > 0$ *and let* $\epsilon > 0$*. Then there exists a quasiconformal map* $f : \mathbb{R}^n \to \mathbb{R}^n$ *(if* $n = 1$*, quasisymmetric) so that* $\dim f(E) > n - \epsilon$*.*

Sketch of the proof:

We already showed this in the case $E = \mathbb{R} \times \{0\}$. The proof in the general case reduces to the following sequence of claims:

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Theorem (Tukia–Väisälä extension theorem, 1982)

Identify $\mathbb{R}$ with $\mathbb{R} \times \{0\} \subset \mathbb{R}^n$. For each $f : \mathbb{R} \to \mathbb{R}$ qs there exists $F : \mathbb{R}^n \to \mathbb{R}^n$ qc s.t. $F|_{\mathbb{R}} = f$.

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(1) Gehring–Väisälä theorem: K-dependent bounds for the distortion of Hausdorff dimension of a fixed set E under a K-qc map of $\mathbb{R}^n$.

(2) Global quasiconformal dimension $GQC\dim(E)$ of a set $E \subset \mathbb{R}^n$: infimum of Hausdorff dimensions of all images of E under a qc map of $\mathbb{R}^n$.

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Next lecture: quasisymmetric maps of general metric spaces and Pansu's conformal dimension.