Distortion of dimension by Sobolev and quasiconformal mappings

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Introduction

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- *Conformal dimension* and *global quasiconformal dimension*: minimize the dimension of a space X among all quasisymmetric images, or of a subset $E \subset X$ among all quasiconformal self-maps of X.
- *Exceptional sets for Sobolev dimension distortion*: quantify the size of certain collections of subsets of a given space X on which a Sobolev mapping can raise the dimension by a definite amount.

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Throughout these lectures, *dimension* will refer (primarily) to *Hausdorff dimension* and (to a lesser extent) to *Assouad dimension*.

Outline of the lecture series

I. Introduction and overview, quasiconformal maps of $\mathbb{R}^n$ and their effect on Hausdorff dimension

II. Global quasiconformal dimension in $\mathbb{R}^n$

III. Conformal dimension of metric spaces

IV. Sobolev dimension distortion in $\mathbb{R}^n$ and in metric spaces

V. QC and Sobolev dimension distortion in the sub-Riemannian Heisenberg group

The slides of the lectures will be posted daily at

`http://www.math.uiuc.edu/~tyson/conferences.html`

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For historical reasons, we begin by reviewing classical results on the distortion of Hausdorff dimension by Euclidean quasiconformal mappings.

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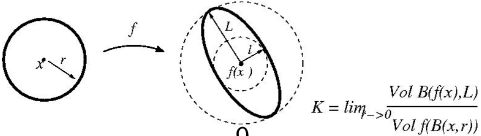
However, we will take advantage of subsequent developments, particularly in the metric theory of quasisymmetric maps, to simplify and streamline some arguments.

We adopt the traditional analytic definition of a quasiconformal mapping.

Definition

A homeomorphism $f : \Omega \to \Omega'$ between domains in $\mathbb{R}^n$ ($n \geq 2$) is *K-quasiconformal (qc)* if $f \in W^{1,n}_{loc}(\Omega : \mathbb{R}^n)$ and $||Df||^n \leq K \det Df$ a.e.



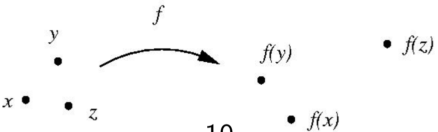
Quasiconformality and quasisymmetry (metric definitions)

Quasiconformal maps have been defined as Sobolev maps whose differential satisfies uniform distortion estimates. Remarkably, the regularity assumption can be dispensed with; such maps can be defined *a priori* as homeomorphisms satisfying a corresponding uniform infinitesimal metric distortion estimate. The required Sobolev regularity can be deduced.

It is convenient, especially in the metric space setting, to require uniform global metric distortion estimates.

Definition (Tukia–Väisälä)

A homeomorphism $f : X \to Y$ between metric spaces is *η-quasisymmetric*, where $\eta : [0, \infty) \to [0, \infty)$ is a homeomorphism, if $|f(x) - f(y)| \leq \eta(t)|f(x) - f(z)|$ whenever $x, y, z \in X$ satisfy $|x - y| \leq t|x - z|$ and $t > 0$.



Every quasisymmetry $f : \Omega \to \Omega'$ between Euclidean domains is trivially quasiconformal according to the so-called *metric definition*: $\exists\, H < \infty$ s.t.

$$\limsup_{r\to 0} \frac{L_f(x,r)}{\ell_f(x,r)} = \limsup_{r\to 0} \frac{\sup\{|f(x)-f(y)| : |x-y| = r\}}{\inf\{|f(x)-f(z)| : |x-z| = r\}} \le H \quad \text{for all } x \in \Omega.$$

Indeed, we can choose $H = \eta(1)$.

Conversely, according to fundamental results of Gehring, each metrically quasiconformal map $f : \Omega \to \Omega'$ is analytically quasiconformal and locally quasisymmetric. When $\Omega = \Omega' = \mathbb{R}^n$, f is in fact (globally) quasisymmetric.

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We emphasize that $n \geq 2$ in this discussion. When $n = 1$ quasiconformality and quasisymmetry are quite distinct notions, for instance, every C^1 diffeomorphism $\mathbb{R} \to \mathbb{R}$ is quasiconformal while, e.g., $x \mapsto x + e^x$ is not quasisymmetric.

Quasiconformal maps: examples and constructions

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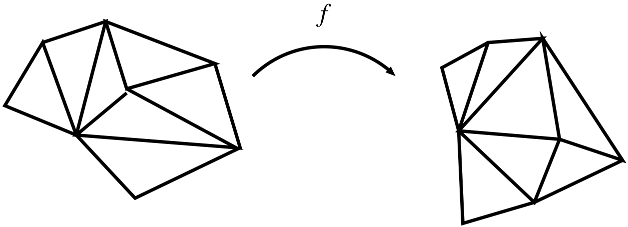
3. PL maps



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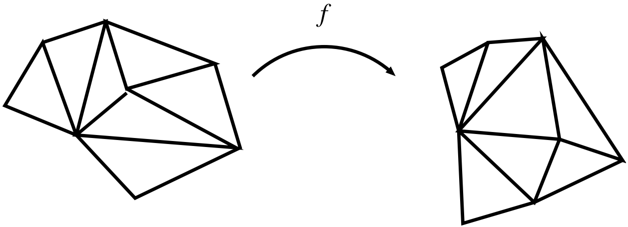

4. QS maps $f : \mathbb{R} \to \mathbb{R}$ correspond to doubling measures μ ($\mu(2I) \leq C\mu(I)$ $\forall$ intervals I):

$$\mu(E) = |f(E)| \quad \leftrightarrow \quad f(x) = \begin{cases} \mu([0,x]) & x > 0, \\ -\mu([x,0]) & x < 0. \end{cases}$$

5. Quasiconformal flows

Reimann (1976) gives conditions on a vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ s.t. the time s map $x(0) \mapsto x(s)$ associated to the Cauchy problem $\dot{x} = V(x)$ is qc. We will not use this technique in the Euclidean setting, however, the corresponding technique in the Heisenberg group is one of the few methods available to construct qc maps.

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6. Quasiconformal gradients

A convex function $u : \mathbb{R}^n \to \mathbb{R}$ is K*-quasiuniformly convex* if $\nabla u : \mathbb{R}^n \to \mathbb{R}^n$ is K-qc. Alternatively, u is K-q.u. convex if $u \in W^{2,n}_{loc}(\mathbb{R}^n)$, u is not affine, and $||D^2 u||^n \leq K \det D^2 u$ a.e., where $D^2 u$ denotes the Hessian of u.

Example

For each $0 < a < 1$, the function $u(x) = \frac{|x|^{1+a}}{1+a}$ is a^{-1}-q.u. convex.

QC gradient maps feature in the proof of Kovalev's theorem on global qc dimension (Lecture II).

Hausdorff measure and dimension

Definition

Let X be a metric space. For $E \subset X$ and $s \geq 0$ the *s-dimensional Hausdorff measure of E* is

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E)$$

where $\mathcal{H}^s_\delta(E) = \inf\{\sum_j (\operatorname{diam} A_j)^s \,:\, E \subset \bigcup_j A_j, \operatorname{diam} A_j \leq \delta\}$.

The *Hausdorff dimension* of E is $\dim E = \inf\{s > 0 \,:\, \mathcal{H}^s(E) = 0\}$.

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Remarks

Spherical Hausdorff measure is defined by replacing covers by arbitrary sets by covers with metric balls. Spherical s-dimensional measure is comparable to $\mathcal{H}^s$, hence the resulting dimension coincides with dim.

When $X = \mathbb{R}^n$ we can instead use coverings by dyadic cubes $Q = \{(y_1, \ldots, y_n) : 2^{-m} j_i \leq y_i \leq 2^{-m}(j_i + 1)\}$, $m, j_1, \ldots, j_n \in \mathbb{Z}$. Note that for dyadic cubes, we can WLOG restrict attention to essentially disjoint covers.

Hausdorff dimensions of Cantor sets

A *contractive similarity* of $\mathbb{R}^n$ is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\exists\, 0 < r < 1$ so that $|f(x) - f(y)| = r|x - y|$ for all $x, y \in \mathbb{R}^n$. Such a map can be written $f(x) = rAx + b$ where $A \in O(n)$ and $b \in \mathbb{R}^n$. r is called the *contraction ratio* of f.

Let $f_1, \ldots, f_M$ be contractive similarities with contraction ratios $r_1, \ldots, r_M$, respectively. Assume that there exists a compact set K so that $f_1(K) \cup \cdots \cup f_M(K) \subset K$ and the sets $f_j(K)$ are pairwise disjoint.

Then there exists a compact set $C \subset K$ s.t. $C = f_1(C) \cup \cdots \cup f_M(C)$ and the sets $f_j(C)$ are pairwise disjoint. Moreover, C is homeomorphic to the standard Cantor set. The Hausdorff dimension s of C is the unique positive solution to the equation

$$\sum_{j=1}^{M} r_j^s = 1.$$

e.g., if $r_1 = \cdots = r_M = r$ then $s = \frac{\log M}{\log 1/r}$.

Example (Gehring–Väisälä)

For any $0 < s < t < n$ there exists $E \subset \mathbb{R}^n$ and a qc map $f : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\dim E = s$ and $\dim f(E) = t$.

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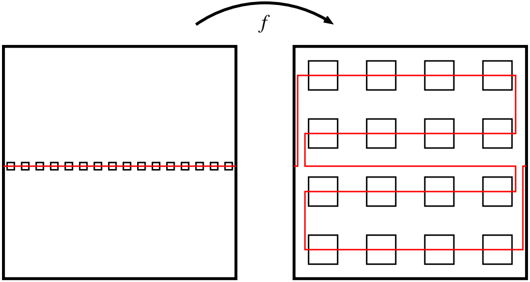

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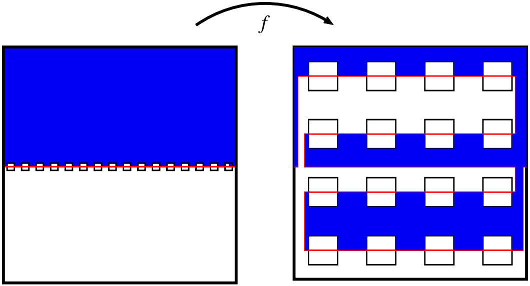

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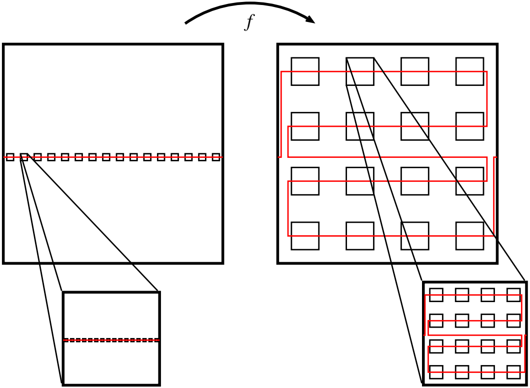

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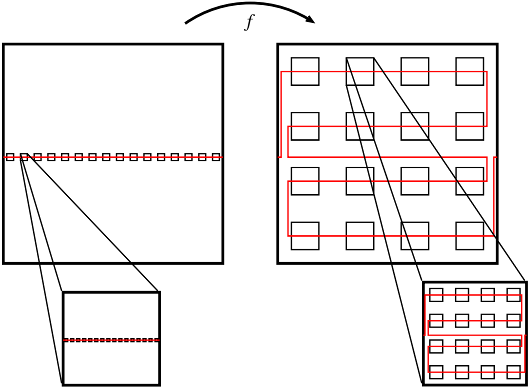

Corollary

Let $E = \mathbb{R} \times \{0\} \subset \mathbb{R}^n$. For each $\epsilon > 0$ there is a qc map f of $\mathbb{R}^n$ s.t. $\dim f(E) > n - \epsilon$.

Dilatation-dependent bounds for qc dimension distortion

Let $n \geq 2$, $K \geq 1$ and $0 < s < n$.

Theorem (Gehring–Väisälä)

There exist constants $0 < \beta = \beta(n, K, s) \leq \alpha(n, K, s) = \alpha < n$ *s.t.*

$$\beta \leq \dim f(E) \leq \alpha$$

whenever $\dim E = s$ *and* f *is* K*-qc in* $\mathbb{R}^n$*. Moreover,* $\alpha(n, K, s) \to 0$ *as* $s \to 0$ *while* $\beta(n, K, s) \to n$ *as* $s \to n$*. In particular,* f *preserves sets of dimension zero and sets of dimension* n*.*

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$$\alpha(n, K, s) = \frac{p(n, K)s}{p(n, K) - n + s}$$

$$p(n, K) = \sup\{p : f \ K\text{-qc in } \mathbb{R}^n \Rightarrow f \in W^{1,p}_{loc}\}.$$

Theorem (Gehring, *Acta Math.*, 1973)

For $n \geq 2$ *and* $K \geq 1$*,* $p(n, K) > n$*.*

Gehring–Väisälä theorem: comments

We will only give the proof of the upper bound $\dim f(E) \leq \alpha(n, K, s)$. The lower bound $\dim f(E) \geq \beta(n, K, s)$ can be obtained by using the fact that the inverse of a quasiconformal map is quasiconformal. (Note that according to the analytic definition which we used, the inverse of a K-qc map may not be K-qc, however, it is K'-qc with $K' = K'(K, n)$.)

A lower bound $\dim f(E) \geq \beta$ could also be obtained just by appealing to the local Hölder continuity of f^{-1}, but this would not give the asymptotic $\beta(n, K, s) \to n$ as $s \to n$.

Gehring–Väisälä theorem: idea of the proof

Starting from a cover $\{B_i\}$ of E by balls, the obvious way to cover $f(E)$ is with the image sets $\{f(B_i)\}$.

Quasiconformality (quasisymmetry) of f implies that the image sets $f(B_i)$ are 'roughly' balls, but does not say anything about the size of $f(B_i)$ relative to the size of B_i. Since Hausdorff dimension is computed by optimizing sums of the form $\sum_i (\operatorname{diam} A_i)^s$, this is potentially a problem.

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Key point: the image sets $f(B_i)$ are roughly balls, in particular,

$$\operatorname{diam} f(B_i) \simeq (\operatorname{Vol}(f(B_i)))^{1/n}.$$

When estimating $\dim f(E)$ via sums of the form $\sum_i (\operatorname{diam} f(B_i))^\alpha$ we can—for a suitable choice of α—reduce to sums of volumes or integrals over these sets.

Via a suitable covering theorem, or by working with dyadic Hausdorff measure, we can ensure that the coverings in question have bounded overlap, yielding finite diameter sums on bounded sets.

Gehring–Väisälä theorem: detailed proof

WLOG $E \subset S \Subset \Omega$. Assume $E \subset \bigcup_j Q_j \subset S$, where the Q_j are essentially disjoint dyadic cubes with diam $Q_j < \delta$. Let $p > n$ s.t. $f \in W^{1,p}_{loc}$.

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$$\mathrm{Vol}\, f(Q_j) = \int_{Q_j} \det Df \leq \int_{Q_j} ||Df||^n \leq (\mathrm{Vol}\, Q_j)^{1-\frac{n}{p}} \left(\int_{Q_j} ||Df||^p \right)^{\frac{n}{p}}$$

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Since $f|_S$ is η-qs for some η, $\exists\, C$ s.t.

$$\operatorname{diam} f(Q_j) \le C(\operatorname{Vol} f(Q_j))^{1/n} \quad \forall\, j.$$

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Since $\mathrm{Vol}\, Q_j \simeq (\mathrm{diam}\, Q_j)^n$, we conclude that

$$\mathrm{diam}\, f(Q_j) \leq C(\mathrm{diam}\, Q_j)^{1-\frac{n}{p}} \left(\int_{Q_j} ||Df||^p \right)^{\frac{1}{p}}.$$

In particular, $\mathrm{diam}\, f(Q_j) < \epsilon = \epsilon(\delta) \to 0$ as $\delta \to 0$.

For $s > \dim E$ set $\alpha = \frac{ps}{p-n+s}$ and compute:

$$\mathcal{H}^{\alpha}_{\epsilon}(f(E)) \leq \sum_j (\mathrm{diam}\, f(Q_j))^{\alpha} \leq C \sum_j (\mathrm{diam}\, Q_j)^{\alpha(1-\frac{n}{p})} \left(\int_{Q_j} ||Df||^p \right)^{\frac{\alpha}{p}}$$

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and taking the infimum over all such covers by cubes and the limit as $\delta \to 0$ gives

$$\mathcal{H}^\alpha(f(E)) \leq C \Big(\mathcal{H}^s(E) \Big)^{1-\frac{\alpha}{p}} ||Df||^\alpha_{L^p(S)} = 0$$

since $s > \dim E$ and $f \in W^{1,p}_{loc}$. Hence $\dim f(E) \leq \alpha$. As $s \searrow \dim E$ and $p \nearrow p(n, K)$, $\alpha \to \alpha(n, K, \dim E)$.

Gehring's higher integrability theorem

f K-qc in $\mathbb{R}^n$, $n \geq 2 \Rightarrow \exists\, p > n$, p depends only on n and K, s.t. $f \in W^{1,p}_{loc}$.

Lemma (Gehring)

Suppose $g \in L^q_{loc}$, $q > 1$, and $\exists\, C$ s.t. $⨍_Q g^q \leq C(⨍_Q g)^q$ for all cubes $Q \subset \mathbb{R}^n$. Then $\exists\, p > q$ and $C' > 0$, both depending only on n, C and q, s.t. $(⨍_Q g^p)^{1/p} \leq C'(⨍_Q g^q)^{1/q}$ for all Q.

We will not discuss Gehring's lemma here. This lemma, and the ideas underpinning its proof, have played a substantial role in many subsequent advances in nonlinear potential theory, PDE and harmonic analysis. An excellent account of the lemma and its significance can be found in

> T. Iwaniec, 'The Gehring Lemma', in *Quasiconformal Mappings and Analysis: A Collection of Papers honoring F. W. Gehring*, Springer, 1998

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Assuming Gehring's lemma, we give a proof of the higher integrability theorem. It suffices to prove that

$$(1) \qquad \fint_Q ||Df||^n \leq C \left(\fint_Q ||Df|| \right)^n \qquad \forall\, Q$$

and then apply Gehring's lemma with $q = n$.
For f qc, $||Df||^n \simeq \det Df$ so (1) is equivalent to

$$\frac{\operatorname{Vol} f(Q)}{\operatorname{Vol}(Q)} \leq C \left(\fint_Q ||Df|| \right)^n .$$

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For a cube $Q \subset \mathbb{R}^n$ of side length s, denote two opposite faces of Q by E and F and let x_Q be the center of Q.
Fix $a \in E$ and $b \in F$. For any $c \in Q$, we may choose $w \in \{a, b\}$ s.t. $|c - w| \geq \frac{1}{2}s$.

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Since f is η-quasisymmetric for some η depending only on n and K,

$$\frac{|f(x_Q) - f(c)|}{|f(a) - f(b)|} \leq \eta(\tfrac{|x_Q - c|}{|w-c|})\eta(\tfrac{|w-c|}{|a-b|}) \leq \eta(\tfrac{\sqrt{n}s}{s/2})\eta(\tfrac{\sqrt{n}s}{s}) \leq C(n, K).$$

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Take the supremum over c and the infimum over a, b:

$$\sup_{c \in Q} |f(x_Q) - f(c)| \leq C(n, K)s' \qquad \text{where } s' := \mathrm{dist}(f(E), f(F)),$$

whence $\mathrm{Vol}\, f(Q) \leq C(n, K)(s')^n$ and so $\frac{\mathrm{Vol}\, f(Q)}{\mathrm{Vol}\, Q} \leq C(n, K)\left(\frac{s'}{s}\right)^n$.

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whence $\operatorname{Vol} f(Q) \leq C(n, K)(s')^n$ and so $\frac{\operatorname{Vol} f(Q)}{\operatorname{Vol} Q} \leq C(n, K)\left(\frac{s'}{s}\right)^n$.
On the other hand, $\int_Q ||Df|| = \int_E (\int_{\gamma_y} ||Df||)\, d\mathcal{L}^{n-1}(y)$ where γ_y denotes the line segment parallel to the remaining side of Q joining $y \in E$ to $z_y \in F$.

$$⨍_Q ||Df|| \geq \frac{1}{\operatorname{Vol}(Q)} \int_E |f(y) - f(z_y)|\, d\mathcal{L}^{n-1}(y) \geq \frac{s'\mathcal{L}^{n-1}(E)}{\operatorname{Vol}(Q)} \geq \frac{(s')s^{n-1}}{s^n} = \frac{s'}{s}.$$

The Gehring–Väisälä theorem quantifies the degree to which the dimension of a set $E \subset \mathbb{R}^n$ can be altered by a K-qc map of $\mathbb{R}^n$.

In the following lecture we will discuss the theory of *global quasiconformal dimension*, which addresses the question: *for a fixed set E, how small can $\dim f(E)$ be if we minimize over all qc maps $f : \mathbb{R}^n \to \mathbb{R}^n$?*

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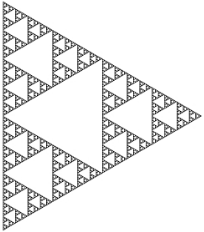
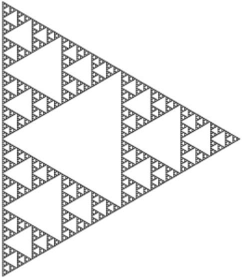
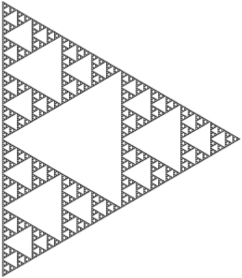
To conclude this first lecture, we answer this question for a classical self-similar fractal, the *Sierpiński gasket SG*.





SG is defined as the *invariant set* $SG = h_0(SG) \cup h_1(SG) \cup h_2(SG)$ associated to three contractive similarities h_0, h_1, h_2 of $\mathbb{R}^2$, each with scaling ratio $\frac{1}{2}$.

$$\dim SG = \frac{\log 3}{\log 2} \approx 1.585\ldots$$



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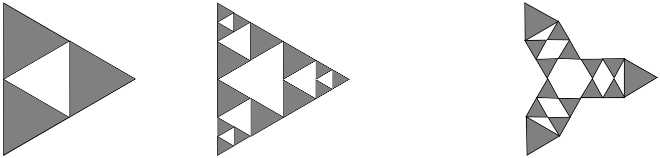
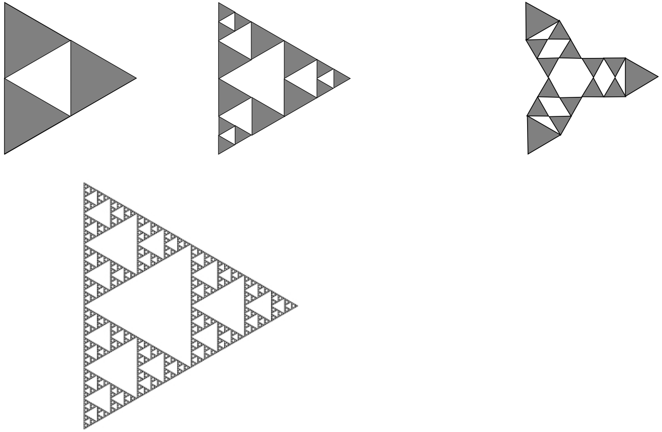
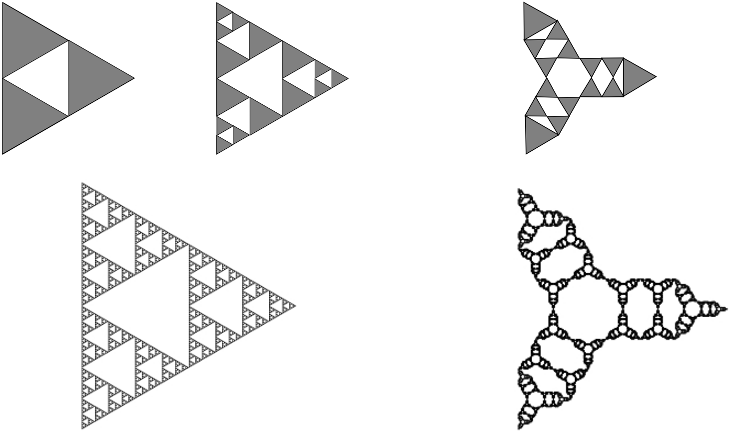
Theorem (T–Wu)

The image of SG under an qc map of $\mathbb{R}^2$ has dimension > 1, and there exist qc maps f_j of $\mathbb{R}^2$ s.t. $\dim f_j(SG) \to 1$.







Theorem (T–Wu)

Let $K, K' \subset \mathbb{R}^n$ be invariant sets for self-similar iterated function systems $\mathcal{F} = \{h_1, \ldots, h_M\}, \mathcal{F}' = \{h'_1, \ldots, h'_M\}$, such that K and K' are canonically homeomorphic. Assume that the IFS $\mathcal{F}$ and $\mathcal{F}'$ satisfy certain geometric assumptions (K and K' are of gasket type*). Then K and K' are canonically quasisymmetrically equivalent.*

Suppose further that $\mathcal{F}$ and $\mathcal{F}'$ are connected by a one-parameter homotopy of self-similar iterated function systems $\mathcal{F}_t$ whose associated invariant sets K_t are pairwise canonically homeomorphic. Then the canonical quasisymmetries $K_t \to K'_t$ can be extended to quasiconformal self-maps of $\mathbb{R}^n$.

Credits

Thanks to Dan Schultz and Nishant Nangia for creating the video of the one-parameter deformation of SG.

Their project was supported by the Illinois Geometry Lab.