Potential Theory in Carnot groups*

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Abstract

In this survey we present a collection of recent results and open problems in potential theory related to the p-Laplacian in connection with Sobolev and Moser – Trudinger inequalities on Carnot groups.

We start with a presentation of the state of art in the Euclidean space. The theory of second-order quasilinear elliptic equations in divergence form is well illustrated by the model equation

$$\triangle_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad 1 < p < \infty. \tag{0.1}$$

This *p-Laplace equation* serves as an L^p-generalization of the classical Laplace equation ($p = 2$), and its solutions, the *p-harmonic functions*, share many properties in common with harmonic functions. For the general theory of equations such as (0.1) and generalizations, we refer to the comprehensive text [28].

The fundamental solution for the *p-Laplace operator* $\triangle_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is explicitly given as a multiple of a suitable power (or logarithm) of the standard Euclidean norm. More precisely, for each $1 < p < \infty$, we have $\triangle_p u_p = c_p\delta_0$, where δ_0 denotes the Dirac distribution at $0 \in \mathbb{R}^n$,

$$u_p(x) = \begin{cases} |x|^{\frac{p-n}{p-1}}, & p \neq n, \\ \log\frac{1}{|x|}, & p = n, \end{cases} \tag{0.2}$$

and

$$c_p = \begin{cases} \operatorname{sign}(n-p)\left(\frac{|n-p|}{p-1}\right)^{p-1}\omega_{n-1}, & p \neq n, \\ \omega_{n-1}, & p = n. \end{cases} \tag{0.3}$$

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Here $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

A critical role in the nonlinear potential theory associated to solutions of (0.1) is played by the classical Sobolev inequalities

$$(0.4) \qquad \left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \le C_p \left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{\frac{1}{p}}, \qquad 1 \le p < n,$$

for u in the Sobolev space $W^{1,p}(\mathbb{R}^n)$. Here $C_p = C_p(\mathbb{R}^n)$ is an absolute (finite) constant; the sharp value for C_p has been computed by Aubin [2] and Talenti [44]:

$$(0.5) \qquad C_p(\mathbb{R}^n) = \begin{cases} \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{1/n}, & p = 1, \\ \frac{1}{n}\left(\frac{n(p-1)}{n-p}\right)^{1-1/p}\left(\frac{\Gamma(n+1)}{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})\omega_{n-1}}\right)^{1/n}, & 1 < p < n. \end{cases}$$

In the borderline case $p = n$ the Sobolev inequality (0.4) is replaced by a condition of exponential integrability. More precisely a result of Trudinger [45] state that there exists $\beta > 0$ so that

$$(0.6) \qquad \sup_{\Omega, u} \frac{1}{|\Omega|} \int_{\Omega} e^{\beta \cdot |u|^{\frac{n}{n-1}}} < \infty,$$

where the supremum is taken over all domains $\Omega \subset \mathbb{R}^n$ and all $u \in W^{1,n}(\Omega)$ with the energy bound $\int_\Omega |\nabla u|^n \le 1$. The sharp coefficient β was determined by Moser [38], who showed that condition (0.6) holds for all relevant Ω and f provided

$$(0.7) \qquad \beta \le \beta_0 = n \cdot \omega_{n-1}^{1/(n-1)},$$

while (0.6) does not hold if $\beta > \beta_0$. For further results related to sharp constants for Trudinger-type inequalities, see Adams [1], Beckner [7], Chang [14], Carleson–Chang [13] and Li [36].

The purpose of this article is to survey recent results on nonlinear potential theory and sharp Sobolev and Trudinger inequalities in the sub-Riemannian environments of general Carnot (stratified Lie) groups. These groups occur as the nilpotent components in the Iwasawa decompositions of semisimple Lie groups. Analysis in these groups is also motivated by their role as models for the general theory of systems of vector fields satisfying Hörmander's condition [48].

We begin with simple examples, the classical *Heisenberg group* and certain generalizations. Capogna, Danielli and Garofalo [12] found explicit formulas for the fundamental solutions to the p-Laplacians which comprise a one-parameter family as in (0.2), while Cohn and Lu [15] computed the sharp constant in the Trudinger inequality on the Heisenberg group. The situation is markedly different in general groups, where the complexity of the Baker–Campbell–Hausdorff formula precludes direct computations. Satisfactory results exist only in the cases $p = 2$ and $p = Q$ (the *homogeneous* or *conformal dimension* of the group). In both of these cases the fundamental solution exists and is unique. Each of these solutions gives rise to a homogeneous norm by the formulas in (0.2), but, in contrast with the Euclidean and Heisenberg cases, these two homogeneous norms may not necessarily be identical (and definitely are not so in at least some situations). Thus two separate and disconnected theories exist, the linear ($p = 2$) theory and the conformal ($p = Q$) theory.

The sharp version of Trudinger's inequality can be established using a representation formula derived from the fundamental solution to the conformal Laplacian $\triangle_Q$. This computation was carried out for general Carnot groups in [4]. Our argument relies on techniques developed by Adams [1] in the Euclidean case and Cohn–Lu [15] in the Heisenberg case. The sharp constants in the other Sobolev inequalities remain unknown—even for the Heisenberg group—with one exception: Jerison and Lee [31] have computed the constant C_2 in (0.4) for the Heisenberg group as a corollary to their solution of the CR Yamabe problem.

The structure of this article is as follows. In section 1 we review the explicit formulas which hold in the Heisenberg group. We then recall the definitions of Kaplan's H-type groups and indicate how the Heisenberg formulas generalize to this case. In section 2 we discuss general Carnot groups, emphasizing those features of the Heisenberg theory which persist even in the absence of explicit formulas. Finally, in section 3 we recall from our previous work [5] the class of *polarizable groups*. This class, which is intermediate between the H-type groups and the general class of Carnot groups, is the largest class for which a one-parameter family of fundamental solutions u_p as in (0.2) exists. While not all Carnot groups are polarizable, we do not know whether any non-H-type polarizable groups exist. The term *polarizable* was attached to these groups in recognition of the fact that they admit a system of *horizontal polar coordinates* which allows us to decompose integrals over the group into "radial" and "spherical" components. Such system of coordinates was introduced by Korányi and Reimann [35] in the setting of the Heisenberg group. We conclude this section with a description of this system in our more general setting. In the final section we summarize some open questions and conjectures.

1 Model cases

1.1 The Heisenberg group

The *Heisenberg group* H is the unique simply connected rank two stratified Lie group of dimension three. We represent this group explicitly as $H = \mathbb{C} \times \mathbb{R}$ with the group operation $(z,t) * (z',t') = (z + z', t + t' + 2\,\mathrm{Im}(z\overline{z'}))$. Denote by X, Y, T the left-invariant vector fields on H which agree with the standard basis $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial t$ at the origin. A computation yields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t},$$

where $z = x + iy$. Thus $[X, Y] = -4T$ and all other Lie brackets are zero. We define a family of dilations $(\delta_\lambda)_{\lambda>0}$ on H which commute with the group multiplication by the formula $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$. The homogenous dimension of the the Heisenberg group is 4, and the function $N(z,t) = (|z|^4 + t^2)^{1/4}$ is a *homogeneous norm* on H, i.e. $N \circ \delta_\lambda = \lambda \cdot N$ for $\lambda > 0$.

We refer to Chapters XII and XIII of [43] for further information on the Heisenberg group, in particular, for a discussion of its roles in several complex variables (analysis on the boundary of strictly pseudoconvex domains in $\mathbb{C}^2$) and in quantum mechanics.

In various differential operators that we are considering, we are allowed to use only the *horizontal vectors* X and Y. Therefore, the analogs of the p-Laplace operators for the Heisenberg group are the *p-sub-Laplace operators* $\triangle_{0,p}$ defined by

$$\triangle_{0,p} f = X(|\nabla_0 f|^{p-2} X f) + Y(|\nabla_0 f|^{p-2} Y f), \qquad f \in C^2(H),$$

where $|\nabla_0 f|^2 = (Xf)^2 + (Yf)^2$ is the square of the norm (related to a scalar product for which $\{X, Y\}$ form an orthonormal basis) of the horizontal gradient given by $\nabla_0 f = (Xf)X + (Yf)Y$. When $p = 2$ this reduces to *Kohn's sub-Laplacian* $\triangle_0 f = XXf + YYf$. In close similarity with the formulas in (1.2) and (0.2). the following remarkable explicit formulas for the fundamental solutions u_p for $\triangle_{0,p}$ are due to Folland [19], [20] in the case $p = 2$ and Capogna–Danielli–Garofalo [12] in the general case.

For each $1 < p < \infty$, we have $-\triangle_{0,p} u_p = c_p \cdot \delta_0$ with

$$u_p(z,t) = \begin{cases} (N(z,t))^{\frac{p-4}{(p-1)}}, & p \neq 4, \\ \log \frac{1}{N(z,t)}, & p = 4, \end{cases} \tag{1.2}$$

and

$$c_p = \begin{cases} \operatorname{sign}(4-p) \left(\frac{|4-p|}{p-1}\right)^{p-1} \frac{\pi^{3/2}\Gamma((p+2)/4)}{2\Gamma((p+4)/4)} & p \neq 4, \\ \pi^2/4, & p = 4. \end{cases} \tag{1.3}$$

The sharp constants $C_p(H)$ and $\beta(H)$ in the Sobolev- and Trudinger-type inequalities

$$\left(\int_H |u|^{\frac{4p}{4-p}}\right)^{\frac{4-p}{4p}} \leq C_p \left(\int_H |\nabla_0 u|^p\right)^{\frac{1}{p}}, \qquad 1 \leq p < 4, \tag{1.4}$$

and

$$\sup_{\substack{\Omega \subset H \\ u \in HW^{1,4}(\Omega), \int_\Omega |\nabla_0 u|^4 \leq 1}} \frac{1}{|\Omega|} \int_\Omega e^{\beta \cdot |u|^{\frac{4}{3}}} \leq C_Q < \infty \tag{1.5}$$

on the Heisenberg group have also recently been a focus of investigation. Cohn and Lu [15] extended an argument of Adams [1] in the Euclidean case to calculate $\beta(H) = 4\pi^{2/3}$. Their argument also makes crucial use of a new representation formula in the Heisenberg setting. In [4] we gave a different derivation of this representation formula using the fundamental solution to the conformal (i.e., the 4-)Laplacian. This argument works in any Carnot group and will be discussed further in section 2.

The only known sharp constant in inequality (1.4) is $C_2(H)$, which was determined by Jerison and Lee as a corollary of their solution to the *CR Yamabe problem* on $S^3 \subset \mathbb{C}^2$. In brief, they show [31] that the only contact forms on S^3 with constant pseudo-Hermitian (Webster) scalar curvature are the images of the standard contact form $\hat{\theta} = \frac{i}{2}(\overline{\partial} - \partial)|z|^2$ under CR automorphisms of S^3. The Cayley transform gives a CR equivalence between H and S^3 minus a single point, and transforms $\hat{\theta}$ to $2u_{JL}^2(dt + 2x\,dy - 2y\,dx)$, where

$$u_{JL}(z,t) = \frac{1}{((|z|^2+1)^2+t^2)^{1/2}}. \tag{1.6}$$

By general results from [30], the minimizers for the Sobolev inequality

$$\left(\int_H |u|^4\right)^{1/4} \le C_2 \left(\int_H |\nabla_0 u|^2\right)^{1/2}$$

are precisely the images of the conformal factor u_{JL} by left translations and dilations on H. An explicit computation gives $C_2(H) = (2\pi)^{-1/2}$.

Observe that the extremal in (1.6) is **not** a function of the homogeneous norm $N(z,t)$. This is in contrast with the conformal case studied by Cohn and Lu, where the extremals for (1.5) are given in terms of this norm (in fact, they are normalized cutoff versions of the basic fundamental solution u_4 in (1.2)). However, Jerison and Lee's extremal function can be expressed using both N and its horizontal gradient:

$$u_{JL} = \frac{1}{(1+2N^2|\nabla_0 N|^2 + N^4)^{1/2}}.$$

1.7 Groups of Heisenberg type

The following definition is due to Kaplan [33].

Definition 1.8. Let G be a simply connected Lie group with stratified Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ satisfying $[V_1, V_1] = V_2$ and $[V_1, V_2] = 0$. Assume that $\mathfrak{g}$ is equipped with an inner product $\langle , \rangle$ and define a linear map $J : V_2 \to \mathrm{End}(V_1)$ by the identity

$$\langle J_Z U, V\rangle = \langle Z, [U,V]\rangle, \qquad U, V \in V_1, Z \in V_2. \tag{1.9}$$

We say that G is of *H(eisenberg)-type* if

$$J_Z^2 = -|Z|^2 \,\mathrm{Id} \tag{1.10}$$

for all $Z \in V_2$, where $|Z|^2 = \langle Z \rangle$.

See [34] and [27, §5] for further information on H-type groups. Cowling et. al. [17] characterized the nilpotent groups arising in the Iwasawa decompositions of the simple rank-one groups $O(n,1)$, $U(n,1)$, $Sp(n,1)$ and F_4^{-20} among the class of H-type groups. There are infinitely many isomorphism classes of groups of Heisenberg type, more precisely, there exists an H-type Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ with $\dim V_1 = k$ and $\dim V_2 = l$ if and only if $l < 8a + 2^b$, where 2^{4a+b} $(0 \le b < 4)$ is the largest power of 2 which divides k [33, Corollary 1]. The Lie algebras of H-type groups are closely related to representations of Clifford algebras, see [18, Section 2(c)] or [39, §7].

Let G be an H-type group. Denote by exp the exponential mapping from $\mathfrak{g}$ to G (since G is simply connected and nilpotent, this is a global diffeomorphism) and define analytic mappings $U : G \to V_1$ and $Z : G \to V_2$ by the identity $x = \exp(U(x) + Z(x))$. Next, let

$$N(x) = (|U(x)|^4 + 16|Z(x)|^2)^{1/4}. \tag{1.11}$$

Then N is a homogeneous norm[1] on G with respect to the dilations $\delta_\lambda : G \to G$ defined by $\delta_\lambda(\exp(U + Z)) = \exp(\lambda U + \lambda^2 Z)$. Denoting $\dim V_1 = k$ and $\dim V_2 = l$ as above, the homogeneous dimension of G is $Q = k + 2l$.

For $1 < p < \infty$, the *p-Laplace operator* is defined in terms of a fixed basis $X_1, \dots, X_k$ for V_1:

$$\triangle_{0,p} f = \sum_{i=1}^{k} X_i(|\nabla_0 f|^{p-2} X_i f), \qquad f \in C^2(G), \tag{1.12}$$

where $|\nabla_0 f|^2 = \sum_i (X_i f)^2$. In light of the discussion in the previous subsection, the following result (due to Kaplan [33] when $p = 2$, Heinonen–Holopainen [27] when $p = Q$, and Capogna–Danielli–Garofalo [12] for general p) is plausible: the fundamental equation $-\triangle_{0,p} u_p = c_p \cdot \delta_0$ on G is satisfied with

$$u_p = \begin{cases} N^{\frac{p-Q}{p-1}}, & p \neq Q, \\ \log \frac{1}{N}, & p = Q, \end{cases} \tag{1.13}$$

and

$$c_p = \begin{cases} \operatorname{sign}(Q-p) \left(\frac{|Q-p|}{p-1} \right)^{p-1} \frac{2\pi^{(k+l)/2}\Gamma((k+p)/4)}{4^l \Gamma(k/2)\Gamma((Q+p)/2)}, & p \neq Q, \\ \frac{2\pi^{(k+l)/2}\Gamma((k+l)/2)}{4^l \Gamma(k/2)\Gamma(Q/2)}, & p = Q. \end{cases} \tag{1.14}$$

Little is known about the values of the sharp constants in the Sobolev and Trudinger inequalities on H-type groups. The only completely known result is for the case of Trudinger's inequality. We can explicitly evaluate the sharp coefficient $\beta = \beta(G)$ in

$$\sup_{\substack{\Omega \subset G \\ u \in HW^{1,Q}(\Omega), \int_\Omega |\nabla_0 u|^Q \le 1}} \frac{1}{|\Omega|} \int_\Omega e^{\beta \cdot |u|^{Q'}} \le C_Q < \infty, \qquad Q' = \frac{Q}{Q-1}; \tag{1.15}$$

the result being $\beta(G) = Q \cdot c_Q^{Q'-1}$, where c_Q is as in (1.14). This is done in [5] by using the polar coordinate decomposition presented in section 3. An independent evaluation of this constant is due to Cohn and Lu [16].

[1] The factor 16 in (1.11) arises because of a normalization mismatch between the definitions of the Heisenberg group and the H-type groups; recall that $[X, Y] = -4T$ in H.

The sharp constant $C_2(G)$ for H-type groups has been considered by Garofalo and Vassilev [23], [24]. Based on the discussion at the end of the previous subsection, it is natural to conjecture that the extremal functions in the Sobolev inequality

$$\left(\int_G |u|^{\frac{2Q}{Q-2}}\right)^{\frac{Q-2}{2Q}} \le C_2 \left(\int_G |\nabla_0 u|^2\right)^{\frac{1}{2}} \tag{1.16}$$

are translates and dilates of the function $u = (1+2N^2|\nabla_0 N|^2+N^4)^{(2-Q)/4}$.[2] In [24], Garofalo and Vassilev verify that this holds true for a certain subclass of functions $u \in HW^{1,2}(G)$ (so-called *partially symmetric* functions) for some H-type groups G. Their proof, following the ideas of Jerison and Lee, consists in studying the solutions to certain Yamabe-type equations.

2 Linear and conformal potential theory in general Carnot groups

According to the discussion of the previous section there is a satisfactory potential theory associated to the p-Laplacian in the Heisenberg or H-type setting. The situation however changes drastically when we pass the realm of H-type groups. In fact, we do not currently know of any non-H-type Carnot groups in which we have fundamental solutions for the p-Laplacians given by explicit formulas such as (1.2) and (1.13). In this section, we describe what is known in general groups. We will see that satisfactory theories can still be developed in the linear and conformal cases. In section 3 we will discuss further the question of when a one-parameter family of solutions as in (1.13) exists.

Let us begin by recalling basic definitions and notation. A *Carnot group* is a simply connected Lie group G with stratified Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$ satisfying $[V_1, V_i] = V_{i+1}$ for all $i = 1, \ldots, r-1$ and $[V_1, V_r] = 0$. We assume that the *horizontal space* V_1 is equipped with an inner product $\langle , \rangle_0$ and a fixed basis $X_1, \ldots, X_k$ which is orthonormal with respect to that inner product. The dilations $\delta_\lambda : G \to G$, $\lambda > 0$, are defined as follows. First, we define linear maps $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ by their action on the basis: $\delta_\lambda(X_i) = \lambda^j X_i$ if $X_i \in V_j$. Then we define $\delta_\lambda : G \to G$ by conjugating with the exponential map: $\delta_\lambda(\exp X) = \exp(\delta_\lambda X)$. The *homogeneous dimension* of G is $Q = \sum_i i \cdot \dim V_i$.

For further discussion on Carnot groups, see [22], [43, Chapter XIII, section 5], [26], [25, Section 11.3] and the references therein. Nonlinear potential theory in Carnot groups has been studied by Heinonen–Holopainen [27], Capogna–Danielli–Garofalo [12] and Capogna [10], [11].

The p-Laplace operators on G are now defined by the formulas in (1.12). In this general setting only two existence results for fundamental solutions to these operators are known:

[2] If true, this would yield the sharp constant

$$C_2(G) = \frac{1}{\sqrt{k(Q-2)}} \left(\frac{2^{k+3l-1}\Gamma(\frac{k+l+1}{2})}{\pi^{(k+l+1)/2}}\right)^{1/Q}.$$

(i) $p = 2$. Folland [21] established the existence and uniqueness of the fundamental solution u_2 for the Laplacian $\triangle_{0,2}$. This solution is smooth (C^∞) away from the identity $0 \in G$ and is homogeneous of order $2 - Q$ ($u_2 \circ \delta_\lambda = \lambda^{2-Q} \cdot u_2$).

(ii) $p = Q$. Heinonen and Holopainen [27] showed the existence of the fundamental solution u_Q for the Q-Laplacian $\triangle_{0,Q}$,[3] and Balogh–Holopainen–Tyson [3] showed that this solution is unique up to an additive constant. The solution is known to be at least Hölder continuous on $G \setminus \{0\}$.

We next discuss the homogeneity properties of the fundamental solution u_Q to the Q-Laplacian. Based on the formulas (1.2) and (1.13), we expect to find that u_Q is not (multiplicatively) homogeneous of a certain order but rather additively homogeneous in the following sense: $u_Q \circ \delta_\lambda = u_Q - c \cdot \log \lambda$ for some $c > 0$. Put another way, we may conjecture that the function

$$N = e^{-a \cdot u_Q} \tag{2.1}$$

is a homogeneous norm ($N \circ \delta_\lambda = \lambda \cdot N$) for some $a = a(G) > 0$. This is in fact the case and was also established in [3]. We briefly indicate the steps involved in the proof. Modulo a few technical details, it suffices to show, for some $c > 0$, that the function $v_\lambda := u_Q \circ \delta_\lambda + c \cdot \log \lambda$ is also a fundamental solution to the Q-Laplace equation and agrees with u_Q at some point $x_0 \neq 0$. Then the uniqueness result cited above guarantees that $v_\lambda = u_Q$. The proof of this result proceeds in two steps. First, we show that there exists some constant $c(\lambda)$ (depending continuously on λ but independent of $x \in G$) so that $\tilde{v}_\lambda(x) := u_Q(\delta_\lambda x) + c(\lambda)$ satisfies the above condition. Consequently $u_Q = u_Q \circ \delta_\lambda + c(\lambda)$. We then show that $c(\lambda) = c \cdot \log \lambda$ for some c. In fact, since the dilation mappings $(\delta_\lambda)_{\lambda > 0}$ form a semigroup, we deduce that $c(\lambda\mu) = c(\lambda) + c(\mu)$, and then observe that the only continuous solutions to this functional equation are multiples of the logarithm.

Thus two separate and independent potential theories have been developed in general Carnot groups. The linear case yields the fundamental solution u_2 and an associated norm $N_2 = u_2^{1/(2-Q)}$, while the conformal case yields the fundamental solution u_Q and an associated norm $N_Q = e^{-a \cdot u_Q}$. Furthermore, we can give examples (a particular example will appear in section 3), where these two norms N_2 and N_Q are not equal.

The Sobolev inequality on general Carnot groups is due to Varopoulos [46], [47]. To our knowledge the problem of best constant in the Sobolev inequality is widely open in this general setting.

The Trudinger inequality on general Carnot groups (with a nonsharp constant β) was first established by Saloff-Coste [40].The sharp coefficient in the Trudinger inequality (1.15) on general groups is a consequence of the results of [3]. The passage from results about

[3]To be precise, it was shown in [27] that a singular solution to the Q-Laplace equation $\triangle_{0,Q} u = 0$ exists in the punctured group $G \setminus \{0\}$ with prescribed behavior at 0 and infinity; that this solution can be properly normalized to become a solution to $-\triangle_{0,Q} u = \delta_0$ throughout G follows by modifying the classical argument of Serrin [41], [42].

fundamental solution for the Q -Laplacian proven in [3] to Trudinger's inequality is as follows. Observe first that the fundamental solution u_Q for the Q-Laplacian induces the formula:

$$f(0) = \int_G |\nabla_0 u_Q(x)|^{Q-2} \langle \nabla_0 u_Q(x), \nabla_0 f(x) \rangle_0 \, dx \ ,$$

for compactly supported, smooth functions $f \in C_0^\infty(G)$. By (2.1) we can replace $u_Q = \frac{1}{a} \log \frac{1}{N}$ which yields

$$f(0) = \frac{-1}{a^{Q-1}} \int_G \frac{|\nabla_0 N(x)|^{Q-2}}{N^{Q-1}(x)} \langle \nabla_0 N(x), \nabla_0 f(x) \rangle_0 \, dx \ .$$

Using group translations we obtain that for $f \in C_0^\infty(G)$ we have the following representations formula:

$$f(y) = \frac{-1}{a^{Q-1}} \int_G \frac{|\nabla_0 N(x)|^{Q-2}}{N^{Q-1}(x)} \langle \nabla_0 N(x), \nabla_0 f(yx^{-1}) \rangle_0 \, dx \ .$$

In [4] we show that the above representation formula implies the sharp constant in Trudinger's inequality. The proof is based on the method of Adams [1] with the essential use of the homogeneity of N. In conclusion we always obtain that $\beta(G) = Q \cdot a(G)$, where $a = a(G)$ is the constant in (2.1).

3 Polarizable groups

It is of interest to determine when we have a one-parameter family of fundamental solutions as in (1.13). In this section we describe an axiom on Carnot groups which suffices for such a result. We then review a few consequences which follow from this fact and conclude with a brief discussion of our current state of knowledge regarding the generality in which our axioms hold. The results in this section are taken from [5].

In order to use our axiom (so-called *polarizability*; for the definition, see below) to establish that a one-parameter family of fundamental solutions exists, we will argue by direct calculation involving the second-order operators $\triangle_{0,p}$. This will require that our solutions be (at least) C^2-smooth away from 0. For this reason, it is natural to begin our discussion with Folland's solution u_2 rather than Heinonen and Holopainen's solution u_Q from the previous section, since the former is known to be C^∞ in the complement of 0.

We will state our axiom in terms of the homogeneous norm $N = u_2^{1/(2-Q)}$ so we begin by discussing what PDE we expect N to satisfy. From the expressions in (1.13) we see that N should satisfy the "$p = \infty$" version of the p-Laplace equation. What is the natural limit of the p-Laplace operators as $p \to \infty$? To find out, we assume that we are given a domain $\Omega \subset G$, a smooth function F on $\partial\Omega$, and a family of solutions $w_p \in C^2(\Omega)$, $p > 1$, to the equations $\triangle_{0,p} w_p = 0$ with $w_p|_{\partial\Omega} = F$ which converge uniformly in Ω to a limit function w_∞ as $p \to \infty$. Expanding this equation yields

$$-\left(|\nabla_0 w_p|^{p-2} \triangle_{0,2} w_p + \frac{p-2}{2} |\nabla_0 w_p|^{p-4} \langle \nabla_0 |\nabla_0 w_p|^2, \nabla_0 w_p \rangle_0 \right) = 0,$$

i.e.

$$\frac{1}{2}\langle \nabla_0 |\nabla_0 w_p|^2, \nabla_0 w_p\rangle_0 = -\frac{1}{p-2}|\nabla_0 w_p|^2 \triangle_{0,2} w_p.$$

Passing to the limit as $p \to \infty$, we find that $f = w_\infty$ must be a solution to the equation

$$\triangle_{0,\infty} f := \frac{1}{2}\langle \nabla_0 |\nabla_0 f|^2, \nabla_0 f\rangle_0 = 0, \tag{3.1}$$

where $\triangle_{0,\infty}$ denotes the ∞-*Laplace operator*. By analogy with the case $p < \infty$, we call solutions to (3.1) ∞-*harmonic functions.*

Note that there is no weak formulation of (3.1) as in the case of finite p. In fact, non-smooth solutions to (3.1) are usually considered in the viscosity sense; see [8], [32], [37], [9]. For C^2 functions f, we can write $\triangle_{0,\infty} f = \sum_{i,j=1}^k X_i f \cdot X_j f \cdot X_i X_j f$.

A direct calculation shows that the homogeneous norm $N(x) = (|U(x)|^4 + 16|Z(x)|^2)^{1/4}$ in ∞-harmonic in any H-type group. By definition, a Carnot group G is said to be *polarizable* if $N = u_2^{1/(2-Q)}$ is ∞-harmonic in $G \setminus \{0\}$. This class of groups is the largest class for which the expressions in (1.13) yield fundamental solutions to the p-Laplace operators. (This can be proved by direct calculation.)

The reason for the name polarizable comes from a further property which these groups enjoy, namely, a system of "horizontal polar coordinates". This system of polar coordinates was identified by Korányi and Reimann [35] in the Heisenberg group and used to compute explicit values for the capacities of some spherical rings. Holopainen [29] used these polar coordinates to prove a Liouville-type theorem for quasiregular mappings of $\mathbb{R}^3$ to H.

We proceed to describe these horizontal polar coordinates. Denote by S the set of points $x \in G$ for which $N(x) = 1$ and $\nabla_0 N(x) \neq 0$. Next, let $\varphi : (0,\infty) \times S \to G$ denote the flow associated to the (non-autonomous) differential equation

$$x'(s) = \frac{N(x)}{s} \cdot \frac{\nabla_0 N(x)}{|\nabla_0 N(x)|^2}, \qquad x(1) = v \in S.$$

Then it follows that $\{\varphi(s,v) : 0 < s < \infty, v \in S\}$ is a set of full Haar measure in G and each curve $\gamma_v(s) = \varphi(s,v)$ is a horizontal curve (i.e., $\gamma_v'(s) \in V_1$ for all s). By a variation of an argument from [22], we can prove the following polar coordinate decomposition for L^1 integrals on G: there exists a Radon measure $d\sigma$ on S so that for all $f \in L^1(G)$ we have

$$\int_G f(x)\,dx = \int_S \int_0^\infty f(\varphi(s,v)) s^{Q-1}\,ds\,d\sigma(v). \tag{3.2}$$

(Here the integral on the left hand side is taken with respect to the Haar measure on G.) While other polar coordinate formulas are known to hold on general Carnot groups (see, for example, Proposition 1.15 in [22]), (3.2) is the first example of such a formula making use of horizontal curves.

It is instructive to observe the particular form which these curves $\varphi(s,v)$ take in the Heisenberg group. By section 4 of [35], in the case $G = H$ we have that the non-singular set is $S = \{(z,t) : |z|^4 + t^2 = 1, z \neq 0\}$ and

$$\varphi(s,v) = (sze^{-i\frac{t}{|z|^2}\log s}, s^2 t), \qquad v = (z,t) \in S, 0 < s < \infty. \tag{3.3}$$

Furthermore, the measure $d\sigma$ on S is given by $d\sigma = d\alpha\, d\theta$, where S is parametrized as $S = \{(\sqrt{\cos\alpha}\, e^{i\theta}, \sin\alpha) : |\alpha| < \pi/2, 0 \leq \theta < 2\pi\}$.

The logarithmic spiral behavior in (3.3) turns out to be standard in this setting. In section 5 of [5], we show that similar expressions for S, $\varphi(s,v)$ and $d\sigma$ can be given on any H-type group. Then the integration formula (3.2) can be used to evaluate specific integrals over the group. (For example, we used this formula to compute the sharp coefficient in the Trudinger inequality (1.15) as well as the value $C_2(G)$ in the footnote on page 7.)

Note that the curves $\varphi(s,v)$ in general fill out only a set of full measure in G; there is a "singular set" $Z = \{x \in G : \nabla_0 N(x) = 0\}$ which is not covered by any of these curves. (Of course, this singular set plays no role in the polar coordinate integration formula (3.2) since it is of zero measure.) In the Heisenberg group, Z coincides with the t-axis, which is the same as the image under the exponential map $\exp : \mathfrak{h} \to H$ of the center $V_2 = \mathrm{span}\{T\} \subset \mathfrak{h}$. A similar fact holds in all H-type groups. We conjecture that the identity $Z = \exp(V_r)$ holds in any polarizable group, cf. equation (5.4) in [11].

We do not presently know of any non-H-type examples of polarizable groups. However, our formulation of the axiom ensures that this question can be definitively resolved in any case where Folland's solution u_2 is explicitly known. The determination of the fundamental solution for Kohn's sub-Laplacian $\triangle_{0,2}$ is a question of much current interest. Beals, Gaveau and Greiner [6] have given an integral representation for u_2 valid on any two-step Carnot group. (Their proof uses methods of complex Hamiltonian mechanics.) In [5] we considered a particular example of such a group, the *anisotroic Heisenberg group* $H_2(a,b)$, $a,b > 0$. This group is characterized by its Lie algebra $\mathfrak{h}_2(a,b) = \mathrm{span}\{X,Y,Z,W\} \oplus \mathrm{span}\{T\}$ and the commutator relations $[X,Y] = -4aT$, $[Z,W] = -4bT$. Using the representation formula from [6], we proved in [5] that Folland's fundamental solution on $H_2(\frac{1}{2},1)$ takes the form

$$u_2(x,y,z,w,t) = \frac{1}{16\pi^2} \frac{\left(\frac{x^2+y^2}{2} + z^2 + w^2 + \sqrt{(\frac{x^2+y^2}{2} + z^2 + w^2)^2 + t^2}\right)^{1/2}}{\sqrt{(\frac{x^2+y^2}{2} + z^2 + w^2)^2 + t^2}\left(\frac{x^2+y^2}{2} + \sqrt{(\frac{x^2+y^2}{2} + z^2 + w^2)^2 + t^2}\right)^{3/2}}.$$

Then a direct calculation showed that $N_6 = u_2^{-1/4}$ ($Q = 6$ in this example) is **not** an ∞-harmonic function. As a corollary, it follows that the two norms N_2 and N_6 discussed in the previous section do not agree.[4]

Carrying out the same computation using the formulas of Beals–Gaveau–Greiner for arbitrary two-step groups appears to be a computation of significant complexity. It would be a result of significant interest, however, to perform this computation and determine once

[4]If they did, then we would have the one-parameter family of fundamental solutions (1.13) and consequently (by sending $p \to \infty$) we would deduce that the norm was ∞-harmonic.

and for all whether any two-step polarizable groups besides the H-type examples exist. Also, the situation for Carnot groups of rank three or higher is completely open at this time.

4 Conclusion

We conclude this survey by gathering together a few questions and conjectures from earlier points in the article.

1. What are the sharp values of the Sobolev embedding constants C_p in (1.4) for the Heisenberg group? (For $p = 1$ this is equivalent with the *isoperimetric problem* for the sub-Riemannian geometry of the Heisenberg group.) How (if at all) are the extremal functions related to the basic norm $N(z,t) = (|z|^4 + t^2)^{1/4}$?

2. Can we establish existence and uniqueness results for the fundamental solutions to the p-Laplacian on general Carnot groups for $p \neq 2, Q$?

3. Are there other polarizable groups besides the H-type examples? When do we have a one-parameter family of fundamental solutions to the p-Laplacians defined in terms of a homogeneous norm?

4. What (if anything) can we say for systems of vector fields satisfying Hörmander's condition which do not occur as a basis for the horizontal space of some Carnot group? Can we calculate fundamental solutions and sharp constants for Sobolev and Trudinger inequalities in this setting? Are there any "polarizable" examples of this type?

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