SMOOTH QUASIREGULAR MAPS WITH BRANCHING IN $\mathbf{R}^n$

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Abstract. According to a theorem of Rickman, all nonconstant $C^{n/(n-2)}$-smooth quasiregular maps in $\mathbf{R}^n$, $n \geq 3$, are local homeomorphisms. Bonk and Heinonen proved that the order of smoothness is sharp in $\mathbf{R}^3$. We prove that the order of smoothness is sharp in $\mathbf{R}^4$. For each $n \geq 5$ we construct a $C^{1+\epsilon(n)}$-smooth quasiregular map in $\mathbf{R}^n$ with nonempty branch set.

1. Introduction

Recall that a continuous mapping $f : D \to \mathbf{R}^n$ in the Sobolev space $W^{1,n}_{\text{loc}}(D, \mathbf{R}^n)$ is called *K-quasiregular* (briefly, *K-QR*), $K \geq 1$, if

$$|f'(x)|^n \leq K J_f(x), \qquad \text{a.e. } x \in D. \tag{1.1}$$

Here $n \geq 2$, $D \subset \mathbf{R}^n$ is a domain, $|f'(x)|$ is the operator norm of the differential of f, and $J_f(x) = \det f'(x)$ denotes the Jacobian determinant. In the plane, 1-quasiregular maps are precisely analytic functions of a single complex variable.

Quasiregular mappings were first introduced and studied by Yu. G. Reshetnyak [18] under the name "mappings of bounded distortion". A deep theorem of Reshetnyak states that nonconstant quasiregular maps are discrete and open. Quasiregular maps were subsequently developed by Martio, Rickman, Väisälä, and their collaborators [15], [16]. See [19], [20] or [13] for a comprehensive account of the theory.

The *branch set* B_f of a continuous, discrete, and open mapping $f : D \to \mathbf{R}^n$ is the closed set of points in D where f does not define a local homeomorphism. By a theorem of Černavskiĭ [4], [5], the topological dimensions of the branch set and its image satisfy

$$\dim B_f = \dim f(B_f) \leq n - 2.$$

On the other hand, if B_f is not empty, then $\Lambda^{n-2}(f(B_f)) > 0$ by a theorem of Martio, Rickman and Väisälä [16], moreover, $\Lambda^{n-2}(B_f) > 0$ when $n = 2$

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(this is trivial) and when $n = 3$ (a result of Martio and Rickman [15]). Here Λ^r is the r-dimensional Hausdorff measure.

Branch sets of quasiregular mappings may exhibit complicated topological structure and may contain, for example, many wild Cantor sets of classical geometric topology [11], [21], [12], [10].

Quasiregular mappings of $\mathbf{R}^2$ can be smooth without being locally invertible. For example, $f(z) = z^2$ has branch set $B_f = \{0\}$. When $n \geq 3$, sufficiently smooth nonconstant quasiregular mappings are locally homeomorphic. In fact

Theorem 1.2. *Every nonconstant $C^{n/(n-2)}$-smooth quasiregular mapping must be locally invertible when $n \geq 3$.*

We say that a mapping $g = (g_1, \ldots, g_n) : D \to \mathbf{R}^n$ is C^k-smooth, $k \in [0, \infty)$, if each coordinate function g_i is in $C^{[k]}(D)$ and if, for every compact set $F \subset G$, there is a constant $C < \infty$ so that

$$\max_{i=1,\ldots,n} |\partial^\alpha g_i(x) - \partial^\alpha g_i(y)| \leq C|x-y|^{k-[k]}$$

whenever $x, y \in F$ and α is a multi-index with $|\alpha| = [k]$.

Theorem 1.2 is due to Rickman [20, p. 12]; an earlier version is due to Church [6]. Rickman's proof uses the Morse-Sard theorem together with the discreteness and openness of quasiregular maps and the theorem of Martio–Rickman–Väisälä mentioned earlier. In his 1978 ICM address [30], Väisälä asked for the sharp smoothness exponent k for which the branch set of every C^k-smooth quasiregular map in $\mathbf{R}^n$, $n \geq 3$, must be empty.

Recently, Bonk and Heinonen [3] showed that the exponent $n/(n-2)$ is sharp when $n = 3$, and proved refined versions of Theorem 1.2 as well as a theorem of Sarvas [22]:

Theorem 1.3. *For every $\epsilon > 0$ and every integer $d \geq 2$, there exists a $C^{3-\epsilon}$-smooth quasiregular mapping $F : \mathbf{R}^3 \to \mathbf{R}^3$ of degree d whose branch set B_F is homeomorphic to $\mathbf{R}$ and has Hausdorff dimension $3 - \delta(\epsilon)$ with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. The map F has the Hölder property*

$$C^{-1}|x-y|^{3-\epsilon'} \leq |F(x) - F(y)| \leq C|x-y|^{3-\epsilon}, \qquad \forall x, y \in B_F, |x-y| \leq 1,$$

for some $0 < \epsilon' \leq \epsilon$ and $C > 1$.

Theorem 1.4. *Given $n \geq 3$ and $K \geq 1$, there exist constants $\lambda = \lambda(n, K) > 0$ and $\delta = \delta(n, K) > 0$ so that (i) the branch set of every K-quasiregular mapping $f : D \to \mathbf{R}^n$ has Hausdorff dimension at most $n - \lambda$, and (ii) every $C^{n/(n-2)-\delta}$-smooth K-quasiregular mapping $f : D \to \mathbf{R}^n$ is locally invertible.*

We prove that Theorem 1.2 is sharp in $\mathbf{R}^4$.

Theorem 1.5. *For every $\epsilon > 0$ and every integer $d \geq 2$, there exists a $C^{2-\epsilon}$-smooth quasiregular mapping $F : \mathbf{R}^4 \to \mathbf{R}^4$ of degree d whose branch set B_F*

is homeomorphic to $\mathbf{R}^2$ *and has Hausdorff dimension* $4-2\epsilon$. *Moreover, the map* F *has the Hölder property*

$$C^{-1}|x-y|^{2-\epsilon} \leq |F(x)-F(y)| \leq C|x-y|^{2-\epsilon}, \qquad \forall x,y \in B_F, |x-y| \leq 1,$$

for some $C > 1$.

The first step in Bonk and Heinonen's proof of Theorem 1.3 is the construction of a quasiconformal mapping g in $\mathbf{R}^3$ with uniformly expanding behavior on a line L. Then g is approximated off L by a C^∞-smooth quasiconformal mapping G by a smoothing procedure of Kiikka [14]. The map G^{-1} has the correct order of smoothness on $\mathbf{R}^3$; postcomposition with a winding map produces the desired quasiregular map F.

As explained in [3], it is not clear how to construct a quasiconformal mapping g in $\mathbf{R}^n$, $n \geq 4$, which is uniformly expanding on a codimension two subspace. Moreover, the smoothing procedure of Kiikka works in dimensions two and three only. Such approximation of general quasiconformal maps can not exist in dimensions higher than five [24], and is an open problem in dimension four [8].

The branch set for our map F in Theorem 1.5 is the product $\Gamma \times \Gamma$ of an infinite snowflake curve with itself. There is a canonical map f from $\Gamma \times \Gamma$ to $\mathbf{R}^2$, which can be written as the composition $f = f_m \circ f_{m-1} \circ \cdots \circ f_1$ of s-quasisymmetric maps with small s. These are quasisymmetric maps which are locally uniformly well-approximated by similarities. Following an extension process developed by Tukia and Väisälä [27], [31] for s-quasisymmetric maps with small s, we extend the maps f_j, $1 \leq j \leq m$, to quasiconformal maps h_j on $\mathbf{R}^4$. Smoothing off $\Gamma \times \Gamma$ via convolution with a variable kernel (see, e.g., [9]) produces smooth quasiconformal maps H_j, $1 \leq j \leq m$. The composition of a winding map with $H_m \circ H_{m-1} \circ \cdots \circ H_1$ yields the desired quasiregular map F.

In general, convolution does not preserve injectivity or quasiconformality. To obtain the injectivity, quasiconformality, and the correct order of smoothness up to and including $\Gamma \times \Gamma$, convolution must be applied in conjunction with the special constructions of Tukia and Väisälä.

Our method does not apply to $\mathbf{R}^n$, $n \geq 5$, unless there exists an appropriate embedding

$$\underbrace{\Gamma \times \cdots \times \Gamma}_{n-2} \hookrightarrow \mathbf{R}^n$$

for some suitable snowflake curve Γ.

Recent results of Bishop [2] and David–Toro [7] provide snowflake-type embeddings $\mathbf{R}^{n-2} \hookrightarrow \Sigma \subset \mathbf{R}^{n-1}$ via global quasiconformal mappings of $\mathbf{R}^{n-1}$; by the Tukia-Väisälä extension theorem [26] these quasiconformal maps of $\mathbf{R}^{n-1}$ can be extended to quasiconformal maps of $\mathbf{R}^n$. The resulting codimension two snowflake-type surfaces $\Sigma \subset \mathbf{R}^n$ can be realized as the branch sets of $C^{1+\epsilon(n)}$-smooth branched quasiregular maps in $\mathbf{R}^n$, $n \geq 5$. More precisely, we have the following result.

Theorem 1.6. *Given integers $n \ge 5$ and $d \ge 2$, there exists $\epsilon = \epsilon(n) > 0$ (independent of d) and a $C^{1+\epsilon}$-smooth quasiregular map $F : \mathbf{R}^n \to \mathbf{R}^n$ of degree d whose branch set B_F is homeomorphic to $\mathbf{R}^{n-2}$. Moreover*

$$C^{-1}|x-y|^{1+\epsilon} \le |F(x)-F(y)| \le C|x-y|^{1+\epsilon}, \qquad \forall x,y \in B_F, |x-y| \le 1,$$

for some $C(n) > 1$.

In section 2 we recall some preliminary material. In section 3 we introduce a one-parameter family of snowflake surfaces in $\mathbf{R}^4$ which are mutually related by canonical quasisymmetric homeomorphisms. In section 4 we extend these homeomorphisms to quasiconformal maps of $\mathbf{R}^4$, and in section 5 we construct smooth approximations to the resulting maps via convolutions. Finally, section 6 contains the proof of Theorem 1.5 and section 7 contains the proof of Theorem 1.6.

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2. Preliminaries

Notation. We write $|x-y|$ for the distance between points x, y in any metric space, and we write $B(x,r)$ for the open ball centered at x of radius r. We denote by $\mathbf{R}^n$ the n-dimensional Euclidean space and by $e_1, \dots, e_n$ the standard basis of $\mathbf{R}^n$. For $0 \le s \le n$ we write Λ^s for the s-dimensional Hausdorff measure. We reserve the notation "dim" to denote the Hausdorff dimension.

A simplex in $\mathbf{R}^n$ is the closed convex hull of a set of $n+1$ points in general position. We write Δ^0 for the set of vertices of a simplex Δ.

For $x \in \mathbf{R}$ we write $[x]$ for the greatest integer less than or equal to x.

We denote by $C, c, \dots$ various constants whose values may change from line to line.

s-Quasisymmetric maps. An embedding $f : X \to Y$ of metric spaces is called *s-quasisymmetric*, $s > 0$, if f is quasisymmetric and satisfies

$$|f(a) - f(x)| \le (t+s)|f(b) - f(x)|$$

whenever $a, b, x \in X$ with $|a-x| \le t|b-x|$ and $t \le 1/s$. Recall that an embedding $f : X \to Y$ is *quasisymmetric* (for short, *QS*) if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ so that

$$|f(a) - f(x)| \le \eta(t)|f(b) - f(x)|$$

whenever $a, b, x \in X$ with $|a-x| \le t|b-x|$. We also say that f is η–*QS*.

Quasisymmetric maps on the real line were introduced by Beurling and Ahlfors [1] as the boundary functions for quasiconformal homeomorphisms of the upper half plane. A systematic study of quasisymmetric maps in metric spaces was begun by Tukia and Väisälä in [25]. s-Quasisymmetric maps were introduced by Tukia and Väisälä in [27] for the study of the extension problem for quasisymmetric maps.

s-Quasisymmetric maps may be characterized as quasisymmetric maps which are locally uniformly close to similarities. A map $h : \mathbf{R}^p \to \mathbf{R}^n$ is *affine* if it is of the form $h(x) = \lambda B(x) + b$, where $\lambda > 0$, $b \in \mathbf{R}^n$, and B is an $n \times p$ matrix. If h is affine and B is orthogonal, we say that h is a *similarity*; in this case we write $L(h) = \lambda$.

In [27] and [31], Tukia and Väisälä proved the following theorems.

Theorem 2.1 (Väisälä [31], Theorem 3.1)**.** *Let $1 \le p \le n$, let A be a compact set in $\mathbf{R}^p$, and let $f : A \to \mathbf{R}^n$ be an s-QS map. Then there is a similarity $h : \mathbf{R}^p \to \mathbf{R}^n$ so that*

$$||h - f||_A \le \varkappa(s,p) L(h) \operatorname{diam} A,$$

where $s \mapsto \varkappa(s,p)$ is an increasing function with $\varkappa(s,p) \to 0$ as $s \to 0$.

Theorem 2.2 (Väisälä [31], Theorem 3.9)**.** *Let $1 \le p \le n$ and $0 < \varkappa \le \frac{1}{25}$, let $X \subset \mathbf{R}^p$ be connected, and let $f : X \to \mathbf{R}^n$ be a map such that for every bounded $A \subset X$, there is a similarity $h : \mathbf{R}^p \to \mathbf{R}^n$ with*

$$||h - f||_A \le \varkappa L(h) \operatorname{diam} A.$$

Then f is s-QS, where $s = s(\varkappa) \to 0$ as $\varkappa \to 0$.

Theorem 2.3 (Tukia–Väisälä [27], Theorem 2.6)**.** *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be an embedding, $n \ge 2$. If f is s-QS, then f is K-quasiconformal, where $K = K(s,n) \to 1$ as $s \to 0$. Conversely, if f is K-quasiconformal, then f is s-QS where $s = s(K,n) \to 0$ as $K \to 1$. Moreover, $f(\mathbf{R}^n) = \mathbf{R}^n$.*

The quasisymmetric extension property. A set $A \subset \mathbf{R}^n$ has the *quasisymmetric extension property* (*QSEP*) if there is $s_0 > 0$ so that if $0 < s \le s_0$, then every s-QS $f : A \to \mathbf{R}^n$ has an s_1-QS extension $g : \mathbf{R}^n \to \mathbf{R}^n$, where $s_1 = s_1(s,n,A) \to 0$ as $s \to 0$. See [31, p. 239].

Theorem 2.4 (Tukia-Väisälä [27], Theorem 5.4)**.** *Let $1 \le p < n$ be integers. Then $\mathbf{R}^p$ has the quasisymmetric extension property in $\mathbf{R}^n$. The parameter s_0 may be chosen depending only on n.*

Following Väisälä [31], we say that a set $A \subset \mathbf{R}^p$ is *thick in* $\mathbf{R}^p$ if there are constants $r_0 > 0$ and $\beta > 0$ so that if $0 < r \le r_0$ and $y \in A$, then there is a simplex Δ in $\mathbf{R}^p$ with $\Delta^0 \subset A \cap B(y,r)$ and $\Lambda^p(\Delta) \ge \beta r^p$.

The Cantor ternary set is thick in $\mathbf{R}^1$, while the von Koch snowflake curve is thick in $\mathbf{R}^2$ (compare Proposition 3.3(a)). Thickness is not bi-Lipschitz invariant.

Theorem 2.5 (Väisälä [31], Theorem 6.2)**.** *Suppose that A is closed and thick in $\mathbf{R}^p$, $1 \le p \le n$, and that either A or $\mathbf{R}^p \setminus A$ is bounded. Then A has the quasisymmetric extension property in $\mathbf{R}^n$. Moreover, $s_0 = s_0(A,n)$ depends only on n,* $\operatorname{diam}(\partial A)$ *and the thickness parameters r_0 and β.*

Whitney triangulations. Let A be a closed, nonempty, proper subset of $\mathbf{R}^n$, and let $\mathcal{K}$ be a Whitney decomposition of $\mathbf{R}^n \setminus A$ into closed dyadic n-cubes (see, e.g., [23, p. 16]). Following [31, p. 253], we define a triangulation $\mathcal{W}$ of $\mathcal{K}$ as follows. Let $\mathcal{W}^0 = \mathcal{K}^0$ consist of all vertices of $\mathcal{K}$. Suppose that a simplicial subdivision $\mathcal{W}^p$ of the p-skeleton $\mathcal{K}^p$ of $\mathcal{K}$ is given. Let Q be a $(p+1)$-cube of $\mathcal{K}$, and let v_Q be the center of Q. Since ∂Q is the underlying space of a subcomplex L_Q of $\mathcal{W}^p$, the cone construction $v_Q L_Q$ gives a simplicial subdivision of Q, and defines $\mathcal{W}^{p+1}$.

The complex $\mathcal{W}$ is called a *Whitney triangulation* of $\mathbf{R}^n \setminus A$.

Remark 2.6. We assume, as we may, that

$$\frac{1}{9} \le \frac{\operatorname{diam} Q}{\operatorname{dist}(Q, A)} \le \frac{1}{4}$$

for all $Q \in \mathcal{K}$. Under this assumption, and by the construction, the simplices of $\mathcal{W}$ belong to a finite number of similarity classes. Therefore there exists a constant $C_1 > 1$ so that each n-simplex σ in $\mathcal{W}$ contains a ball of radius $C_1^{-1} \operatorname{diam} \sigma$.

Regularized distance functions. Let A be a closed, nonempty, and proper subset of $\mathbf{R}^n$, and let $\mathcal{W}$ be a Whitney triangulation of $\mathbf{R}^n \setminus A$. Let δ_A be a positive C^∞-smooth function on $\mathbf{R}^n \setminus A$ so that

$$\frac{1}{10^4 C_1} \le \frac{\delta_A(x)}{\operatorname{dist}(x, A)} \le \frac{1}{10^2 C_1}, \tag{2.7}$$

$$\left| \frac{\partial \delta_A}{\partial x_j}(x) \right| \le C_2, \tag{2.8}$$

and

$$\left| \frac{\partial^2 \delta_A}{\partial x_i \partial x_j}(x) \right| \le \frac{C_2}{\delta_A(x)} \tag{2.9}$$

for all $x \notin A$, where C_1 is the value from Remark 2.6 and C_2 is a constant depending only on n. See, for example, [23, p. 170].

Remark 2.10. Each n-simplex $\sigma \in \mathcal{W}$ contains a ball centered at some point x_σ of radius $2\delta_A(x_\sigma)$.

Smoothing with a variable kernel. Fix a real valued function φ in $C^\infty(\mathbf{R}^n)$ which is nonnegative, radial, supported in $B(0,1)$, and satisfies $\int_{\mathbf{R}^n} \varphi(x)\, dx = 1$ and

$$\sup_{\mathbf{R}^n} \left| \frac{\partial \varphi}{\partial x_i} \right|, \sup_{\mathbf{R}^n} \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right| \le C_3 \tag{2.11}$$

for some C_3 depending at most on n.

Lemma 2.12. *Let $f(x) = Bx + b$ be an affine map with $B \in \mathbf{R}^{n\times n}$ and $b \in \mathbf{R}^n$. Then*

$$f(x) = \int_{\mathbf{R}^n} f(y) \varphi(x - y)\, dy.$$

Proof. Since φ is radial,

$$\int_{\mathbf{R}^n} (By+b)\varphi(x-y)\,dy = \int_{\mathbf{R}^n} (-By+Bx+b)\varphi(y)\,dy = Bx+b.$$

□

Proposition 2.13. *Let A be a closed, nonempty, proper subset of $\mathbf{R}^n$, and let $\delta \equiv \delta_A$ be the regularized distance function on $\mathbf{R}^n \setminus A$ from the previous paragraph. Let u be a real-valued function on $\mathbf{R}^n$, and denote by $\mathrm{Osc}(u,x,r)$ the oscillation of u on $B(x,r)$. Set*

$$U(x) = \begin{cases} \frac{1}{\delta^n(x)} \int_{\mathbf{R}^n} u(y)\varphi(\frac{x-y}{\delta(x)})\,dy, & x \in \mathbf{R}^n \setminus A, \\ u(x), & x \in A. \end{cases}$$

Then

(i) *U is C^∞ on $\mathbf{R}^n \setminus A$;*
(ii) *if u is continuous, then U is continuous;*
(iii) *for $x \in \mathbf{R}^n \setminus A$,*

$$\left|\frac{\partial U}{\partial x_j}(x)\right| \le \frac{C_4}{\delta(x)} \mathrm{Osc}(u,x,\delta(x)) \tag{2.14}$$

and

$$\left|\frac{\partial^2 U}{\partial x_i \partial x_j}(x)\right| \le \frac{C_5}{\delta^2(x)} \mathrm{Osc}(u,x,\delta(x)), \tag{2.15}$$

where C_4 and C_5 are constants depending only on n.

Smoothing by convolution with a variable kernel has occurred in the literature, cf. Arakelyan's approximation theorems [9].

The proof of Proposition 2.13 is by direct calculation and is omitted.

3. Quasisymmetrically equivalent snowflake surfaces in $\mathbf{R}^4$

In this section, we consider snowflake surfaces $\Gamma^\alpha \times \Gamma^\alpha$ in $\mathbf{R}^4$, $1 \le \alpha < 2$, where Γ^α is the periodic extension of a standard von Koch snowflake segment γ^α in $\mathbf{R}^2$. For each $\alpha \in [1,2)$, $\Gamma^\alpha \times \Gamma^\alpha$ is canonically homeomorphic with $\Gamma^1 \times \Gamma^1 = \mathbf{R}^2$. We show that this homeomorphism is quasisymmetric, and factors as a composition of s-quasisymmetric maps. Moreover, $\Gamma^\alpha \times \Gamma^\alpha$ is thick in $\mathbf{R}^4$ for each $\alpha > 1$, with parameters $r_0 \equiv 1$ and $\beta = \beta(\alpha) \searrow 0$ as $\alpha \to 1$.

von Koch snowflake curves in $\mathbf{R}^2$. Fix $1 \le \alpha < 2$, and define γ^α to be the von Koch-type snowflake curve, homeomorphic with $\gamma^1 = [0,1]$, consisting of four self-similar pieces scaled by factor

$$r = r(\alpha) = 4^{-1/\alpha} \in [1/4, 1/2).$$

Precisely, let

$$\theta = 2\arccos(2^{-1+1/\alpha}) \in [0, \pi/2),$$

(note that $r + r\cos\theta = \frac{1}{2}$), and define contractive similarities $\varphi_i^\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\begin{aligned} \varphi_1^\alpha = \delta_r, \quad \varphi_2^\alpha = \delta_r(e_1 + R_\theta), \\ \varphi_3^\alpha = \rho\delta_r(e_1 + R_{-\theta})\rho, \quad \varphi_4^\alpha = \rho\delta_r\rho, \end{aligned} \tag{3.1}$$

where $\delta_r : \mathbf{R}^2 \to \mathbf{R}^2$, $R_\theta : \mathbf{R}^2 \to \mathbf{R}^2$, and $\rho : \mathbf{R}^2 \to \mathbf{R}^2$ are given by

$$\delta_r(x_1, x_2) = (rx_1, rx_2), \quad r > 0,$$

$$R_\theta(x_1, x_2) = (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta), \quad 0 < \theta < 2\pi,$$

and

$$\rho(x_1, x_2) = (1 - x_1, x_2).$$

The von Koch snowflake curve γ^α is the invariant set for the iterated function system $\mathcal{F}^\alpha = \{\varphi_1^\alpha, \varphi_2^\alpha, \varphi_3^\alpha, \varphi_4^\alpha\}$, i.e., the unique compact subset of $\mathbf{R}^2$ verifying

$$\gamma^\alpha = \bigcup_{i=1}^4 \varphi_i^\alpha(\gamma^\alpha).$$

Since $\mathcal{F}^\alpha$ satisfies the open set condition [17],

$$\dim\gamma^\alpha = \frac{\log 4}{\log 1/r} = \alpha.$$

Denote by

$$\Gamma^\alpha = \cup_{n\in\mathbf{Z}}(\gamma^\alpha + n \cdot e_1)$$

the equivariant extension of γ^α with respect to the action of $\mathbf{Z}$ on $\mathbf{R}^2$ by translation in the first coordinate.

We write $S = \{1, 2, 3, 4\}$, and we denote by $\Sigma = S^{\mathbf{N}}$, respectively S^*, the space of all infinite, respectively finite, words with letters drawn from S. When Σ is endowed with the product topology arising from the discrete topology on S, the map $\pi^\alpha : \Sigma \to \gamma^\alpha$ given by

$$\pi^\alpha(w) = \lim_{m\to\infty} \varphi_{w_1}^\alpha \circ \cdots \circ \varphi_{w_m}^\alpha(0), \qquad w = (w_1, \ldots, w_m \ldots),$$

becomes a continuous map of compact sets. Thus the maps

$$f_\alpha^{\alpha'} := \pi^{\alpha'} \circ (\pi^\alpha)^{-1}$$

are well-defined homeomorphisms from γ^α to $\gamma^{\alpha'}$, $1 \le \alpha, \alpha' < 2$. We call $f_\alpha^{\alpha'}$ the *canonical homeomorphism* from γ^α to $\gamma^{\alpha'}$. Observe that $f_\alpha^{\alpha'}$ extends to a homeomorphism of Γ^α onto $\Gamma^{\alpha'}$ which is equivariant with respect to the group action:

$$f_\alpha^{\alpha'}(x_1, x_2) = ([x_1], 0) + f_\alpha^{\alpha'}(x_1 - [x_1], x_2).$$

For $w = (w_1, \ldots, w_m) \in S^*$ we let $\varphi_w^\alpha := \varphi_{w_1}^\alpha \circ \cdots \circ \varphi_{w_m}^\alpha$. We call the sets

$$\varphi_w^\alpha(\gamma^\alpha) + n \cdot e_1, \quad w \in S^*, n \in \mathbf{Z},$$

the 4-*adic similarity pieces* of Γ^α.

Snowflake surfaces in $\mathbf{R}^4$. Consider the product sets $\gamma^\alpha \times \gamma^\alpha$ and $\Gamma^\alpha \times \Gamma^\alpha$ in $\mathbf{R}^4$. Define
$$F_\alpha^{\alpha'} = f_\alpha^{\alpha'} \times f_\alpha^{\alpha'} : \Gamma^\alpha \times \Gamma^\alpha \to \Gamma^{\alpha'} \times \Gamma^{\alpha'}$$
and note that the image of $\gamma^\alpha \times \gamma^\alpha$ under $F_\alpha^{\alpha'}$ is $\gamma^{\alpha'} \times \gamma^{\alpha'}$.

Remark 3.2. Observe that $\Gamma^\alpha \times \Gamma^\alpha$ and $F_\alpha^{\alpha'}$ are equivariant with respect to the action of $\mathbf{Z} \times \mathbf{Z}$ on $\mathbf{R}^4$ by translation in the first and third coordinates. That is,
$$\Gamma^\alpha \times \Gamma^\alpha + (i,0,j,0) = \Gamma^\alpha \times \Gamma^\alpha$$
and
$$F_\alpha^{\alpha'}(x_1+i, x_2, x_3+j, x_4) = (i,0,j,0) + F_\alpha^{\alpha'}(x_1,x_2,x_3,x_4)$$
for all $(i,j) \in \mathbf{Z} \times \mathbf{Z}$ and all $(x_1,x_2,x_3,x_4) \in \mathbf{R}^4$. For the remainder of the paper, we use the term "equivariant" to refer to this specific group action.

Our goal in this section is to prove the following proposition.

Proposition 3.3. (a) *For each $\alpha \in (1,2)$, $\Gamma^\alpha \times \Gamma^\alpha$ is thick in $\mathbf{R}^4$ with parameters $r_0 \equiv 1$ and $\beta(\alpha) \to 0$ as $\alpha \to 1$.*

(b) *For each $\alpha, \alpha' \in [1,2)$, there exists $C = C(\alpha,\alpha') < \infty$ so that*

$$C^{-1}|x-y|^{\alpha/\alpha'} \le |F_\alpha^{\alpha'}(x) - F_\alpha^{\alpha'}(y)| \le C|x-y|^{\alpha/\alpha'} \tag{3.4}$$

for all $x, y \in \Gamma^\alpha \times \Gamma^\alpha$, $|x-y| \le 1$.

Let $s_0(\alpha)$ be a positive function defined on $[1,2)$.

(c) *There exists $\epsilon(\alpha) > 0$ so that for each $\alpha' \in (\alpha - \epsilon(\alpha), \alpha + \epsilon(\alpha))$ (if $\alpha > 1$) or $\alpha' \in [1, 1+\epsilon(1))$ (if $\alpha = 1$), the canonical homeomorphism $F_\alpha^{\alpha'}$ is s-quasisymmetric, with $s < s_0(\alpha)$ and $s \to 0$ as $\alpha' \to \alpha$.*

(d) *For each $\alpha \in (1,2)$, there is a finite sequence*
$$1 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = \alpha$$
so that for each k, either $F_{\alpha_k}^{\alpha_{k+1}}$ is $s_0(\alpha_k)$-QS or $(F_{\alpha_k}^{\alpha_{k+1}})^{-1}$ is $s_0(\alpha_{k+1})$-QS.

In the proof of Proposition 3.3(c) we use the following lemmas. For any $n \ge 1$, we denote by id the identity map on $\mathbf{R}^n$.

Lemma 3.5. *$||\operatorname{id} - f_\alpha^{\alpha'}||_{\Gamma^\alpha} \to 0$ and $||\operatorname{id} - F_\alpha^{\alpha'}||_{\Gamma^\alpha \times \Gamma^\alpha} \to 0$ as $\alpha' \to \alpha$.*

Proof. This lemma is obvious; we include a sketch of the proof for completeness. In [28] we introduced a metric D on iterated function systems on Euclidean spaces. From the definition of the maps φ_i^α in (3.1), it is straightforward to verify that $D(\mathcal{F}^\alpha, \mathcal{F}^{\alpha'}) \to 0$ as $\alpha' \to \alpha$. From Remark 5.35 of [28] we deduce that
$$||\operatorname{id} - f_\alpha^{\alpha'}||_{\gamma^\alpha} \to 0 \quad \text{as } \alpha' \to \alpha,$$
whence
$$||\operatorname{id} - F_\alpha^{\alpha'}||_{\gamma^\alpha \times \gamma^\alpha} \to 0 \quad \text{as } \alpha' \to \alpha.$$
The lemma follows by the equivariance of $f_\alpha^{\alpha'}$ and $F_\alpha^{\alpha'}$. □

Lemma 3.6. *Let $\alpha,\alpha'\in[1,2)$, and assume that $J\subset\Gamma^\alpha$ is either a 4-adic similarity piece of Γ^α or the union of two adjacent 4-adic similarity pieces of generation m. Then there is a similarity $h_J:\mathbf{R}^2\to\mathbf{R}^2$ with*

$$L(h_J)=4^{-(1/\alpha'-1/\alpha)m} \tag{3.7}$$

so that

$$||f_\alpha^{\alpha'}-h_J||_J\le\varkappa L(h_J)\operatorname{diam}J, \tag{3.8}$$

where $\varkappa(\alpha,\alpha')\to0$ as $\alpha'\to\alpha$.

Proof of Proposition 3.3. Part (a) is easy to prove and will be omitted. To prove (b), it suffices to show that

$$|f_\alpha^{\alpha'}(x)-f_\alpha^{\alpha'}(y)|\simeq|x-y|^{\alpha/\alpha'} \tag{3.9}$$

for fixed $\alpha,\alpha'\in[1,2)$ and all $x,y\in\Gamma^\alpha$, $|x-y|\le1$. The notation $A\simeq B$ means that there exists a constant C, depending at most on α and α', so that $C^{-1}B\le A\le CB$.

We write $f=f_\alpha^{\alpha'}$. Consider the case $x,y\in\gamma^\alpha$, and note by scaling that it suffices to assume that no proper copy $\varphi_i(\gamma^\alpha)$, $i\in S$, contains both x and y. Choose $i,j\in S$ so that $x\in\varphi_i(\gamma^\alpha)$ and $y\in\varphi_j(\gamma^\alpha)$. If $\varphi_i(\gamma^\alpha)$ and $\varphi_j(\gamma^\alpha)$ are disjoint, then $|x-y|\ge1/2-r_\alpha$ and so $|f(x)-f(y)|\ge1/2-r_{\alpha'}$. On the other hand, if $\varphi_i(\gamma^\alpha)$ and $\varphi_j(\gamma^\alpha)$ intersect, let $\{v\}=\varphi_i(\gamma^\alpha)\cap\varphi_j(\gamma^\alpha)$ and note that $|x-y|\simeq\max\{|x-v|,|y-v|\}$. Assuming $|x-v|\ge|y-v|$ and choosing k so that $|x-v|\simeq r_\alpha^k$, we find

$$|f(x)-f(y)|\simeq|f(x)-f(v)|\simeq(r_{\alpha'})^k=r_\alpha^{k\alpha/\alpha'}\simeq|x-v|^{\alpha/\alpha'}\simeq|x-y|^{\alpha/\alpha'}.$$

The case when $x\in\gamma^\alpha$ and $y\in\gamma^\alpha+e_1$ is similar. This completes the proof of (3.9).

Assuming the validity of part (c), we establish (d). For fixed $1<\alpha<2$, the sets $I(1)=[1,1+\epsilon(1))$ and $I(\beta)=(\beta-\epsilon(\beta),\beta+\epsilon(\beta))$, $1<\beta\le\alpha$, form an open cover of $[1,\alpha]$. Choose a finite subcover $I(\beta_1),\dots,I(\beta_N)$ with $1=\beta_1<\dots<\beta_N=\alpha$ and $I(\beta_j)\cap I(\beta_k)\ne\emptyset$ if and only if $|j-k|\le1$. For each $j<N$ with $\beta_{j+1}\notin I(\beta_j)$, choose $\gamma_j\in I(\beta_j)\cap I(\beta_{j+1})$. The desired sequence $\alpha_0,\dots,\alpha_M$ is chosen to consist of the numbers $\beta_0,\dots,\beta_N$, together with all values γ_j above, arranged in increasing order. Note that if $\beta_{j+1}\in I(\beta_j)$, then $F_{\beta_j}^{\beta_{j+1}}$ is $s_0(\beta_j)$-QS, while if $\beta_{j+1}\notin I(\beta_j)$, then $F_{\beta_j}^{\gamma_j}$ is $s_0(\beta_j)$-QS and $(F_{\gamma_j}^{\beta_{j+1}})^{-1}$ is $s_0(\beta_{j+1})$-QS. This proves (d).

It remains to verify part (c). We will show that the hypotheses of Theorem 2.2 hold for $f=F_\alpha^{\alpha'}$ and $X=\Gamma^\alpha\times\Gamma^\alpha$, with $\varkappa\to0$ as $\alpha'\to\alpha$. To simplify the presentation, we do this in the case $\alpha=1$ only; the argument can be easily modified for the case $\alpha>1$.

Thus consider the canonical homeomorphism $F_1^\alpha:\mathbf{R}^2\to\Gamma^{\alpha'}\times\Gamma^{\alpha'}$. Suppose that A is a bounded set in $\mathbf{R}^2$. Choose a square $J_1\times J_2\supset A$ subject to the following constraints:

(i) each side J_i, $i = 1, 2$, is either a 4-adic similarity piece of $\Gamma^1 = \mathbf{R}$ or the union of two such 4-adic similarity pieces of the same generation;
(ii) $\operatorname{diam} J_1 \times J_2 \le 8\sqrt{2} \operatorname{diam} A$.

By Lemma 3.6 there exist similarities $h_{J_1}, h_{J_2} : \mathbf{R}^2 \to \mathbf{R}^2$ with $L(h_{J_1}) = L(h_{J_2}) =: L$ and

$$||f_1^{\alpha'} - h_{J_i}||_{J_i} \le \varkappa L \operatorname{diam} J_i$$

with $\varkappa \to 0$ as $\alpha' \to 1$. The product $h = h_{J_1} \times h_{J_2}$ is a similarity mapping of $\mathbf{R}^4$ with $L(h) = L$, and

$$\begin{aligned} ||F_1^{\alpha'} - h||_{J_1 \times J_2} &\le \sqrt{2} \cdot \max\{||f_1^{\alpha'} - h_{J_1}||_{J_1}, ||f_1^{\alpha'} - h_{J_2}||_{J_2}\} \\ &\le \sqrt{2} \cdot \varkappa L(h) \operatorname{diam}(J_1 \times J_2) \\ &\le 16 \varkappa L(h) \operatorname{diam} A. \end{aligned}$$

By Theorem 2.2, $F_1^{\alpha'}$ is s-QS for α' sufficiently close to one, with $s \to 0$ as $\alpha' \to 1$. In particular, for $\epsilon(1)$ sufficiently small, $F_1^{\alpha'}$ is $s_0(1)$-QS. $\square$

It remains to prove Lemma 3.6.

Proof of Lemma 3.6. We consider two cases:

Case 1: $J = \varphi_w^\alpha(\gamma^1) + ne_1$ for some $w \in S^*$ and $n \in \mathbf{Z}$;
Case 2: $J = (\varphi_{w_L}^\alpha(\gamma^1) + n_L e_1) \cup (\varphi_{w_R}^\alpha(\gamma^1) + n_R e_1)$ for some $w_L, w_R \in S^*$ and $n_L, n_R \in \mathbf{Z}$.

By equivariance, we may assume without loss of generality that $n = 0$ (case 1) or that $n_L = 0$ (case 2). Choose $m \ge 0$ so that J is either a similarity piece or the union of two adjacent similarity pieces of generation m.

First, we consider case 1. Let

$$h_J = \varphi_w^{\alpha'} \circ (\varphi_w^\alpha)^{-1}$$

and observe that $L(h_J)$ is given by the formula in (3.7). Since

$$f_\alpha^{\alpha'} = \varphi_w^{\alpha'} \circ f_\alpha^{\alpha'} \circ (\varphi_w^\alpha)^{-1}$$

for all words $w \in S^*$, we have by Lemma 3.5 that

$$||f_\alpha^{\alpha'} - h_J||_J = r(\alpha')^m ||f_\alpha^{\alpha'} - \operatorname{id}||_{\gamma^\alpha} \le c(\alpha, \alpha') r(\alpha')^m$$

with $c(\alpha, \alpha') \to 0$ as $\alpha' \to \alpha$. Thus

$$||f_\alpha^{\alpha'} - h_J||_J \le \varkappa 4^{-(1/\alpha' - 1/\alpha)m} \operatorname{diam} J \tag{3.10}$$

with $\varkappa \to 0$ as $\alpha' \to \alpha$. (Recall that $r(\alpha') = 4^{-1/\alpha'}$ and observe that $\operatorname{diam} J \simeq r(\alpha)^m = 4^{-m/\alpha}$.)

Next we consider case 2. Let $J_L = \varphi_{w_L}^\alpha(\gamma^1) + n_L e_1$ and $J_R = \varphi_{w_R}^\alpha(\gamma^1) + n_R e_1$. We distinguish two subcases: (i) $n_R = 0$, and (ii) $n_R = 1$. In case 2(i), let

$$h_J = \varphi_{w_L}^{\alpha'} \circ (\varphi_{w_L}^\alpha)^{-1}. \tag{3.11}$$

The estimates

$$||f_\alpha^{\alpha'} - h_J||_{J_L} \le c(\alpha, \alpha') 4^{-(1/\alpha' - 1/\alpha)m} \operatorname{diam} J$$

and

$$||f_\alpha^{\alpha'} - \varphi_{w_R}^\alpha \circ (\varphi_{w_R}^\alpha)^{-1}||_{J_R} \le c(\alpha,\alpha') 4^{-(1/\alpha' - 1/\alpha)m} \operatorname{diam} J$$

hold as in case 1. By the estimate in [28, (5.23)],

$$||\varphi_{w_L}^{\alpha'} \circ (\varphi_{w_L}^\alpha)^{-1} - \varphi_{w_R}^{\alpha'} \circ (\varphi_{w_R}^\alpha)^{-1}||_{J_R} \le c'(\alpha,\alpha') 4^{-(1/\alpha' - 1/\alpha)m} \operatorname{diam} J$$

with $c'(\alpha,\alpha') \to 0$ as $\alpha' \to \alpha$. Thus (3.10) holds for a suitable choice of $\varkappa$.

In the final case 2(ii), define h_J as in (3.11). By equivariance,

$$||\varphi_{w_L}^{\alpha'} \circ (\varphi_{w_L}^\alpha)^{-1} - (1+\varphi_{w_R}^{\alpha'}) \circ (1+\varphi_{w_R}^\alpha)^{-1}||_{J_R} \le c'(\alpha,\alpha') 4^{-(1/\alpha' - 1/\alpha)m} \operatorname{diam} J$$

with $c'(\alpha,\alpha') \to 0$ as $\alpha' \to \alpha$, and again we obtain the inequality in (3.10) for a suitable $\varkappa$. This completes the proof of Lemma 3.6. □

4. Quasiconformal extension of equivariant s-quasisymmetric maps

For $\alpha \in [1,2)$ we consider the snowflake surface $\Gamma^\alpha \times \Gamma^\alpha$ from the previous section. Set

$$I = [0,1] \times [-2,2] \times [0,1] \times [-2,2],$$

$$J = \mathbf{R} \times (-1,1) \times \mathbf{R} \times (-1,1),$$

and

$$A^\alpha = \Gamma^\alpha \times \Gamma^\alpha \cap ([-4,4] \times \mathbf{R} \times [-4,4] \times \mathbf{R}).$$

Observe that $\Gamma^\alpha \times \Gamma^\alpha \subset \mathbf{R} \times [0, \sqrt{2}/2] \times \mathbf{R} \times [0, \sqrt{2}/2]$ and that $A^\alpha \subset B(0,6)$.

For each $\alpha \in [1,2)$ we fix a Whitney triangulation $\mathcal{W}^\alpha$ of $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ and a Whitney triangulation $\widetilde{\mathcal{W}}^\alpha$ of $\mathbf{R}^4 \setminus A^\alpha$ constructed by the procedures in section 2. We require in addition the following:

- The interior of any n-simplex in $\mathcal{W}^\alpha$ or $\widetilde{\mathcal{W}}^\alpha$ does not meet ∂I;
- (Common triangulations for $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ and $\mathbf{R}^4 \setminus A^\alpha$ near the origin) For each $\alpha \in [1,2)$ and each n-simplex $\sigma \subset I$, we have $\sigma \in \mathcal{W}^\alpha$ if and only if $\sigma \in \widetilde{\mathcal{W}}^\alpha$.
- (Equivariant simplices) For each $\alpha \in [1,2)$, if $\sigma \in \mathcal{W}^\alpha$ is an n-simplex with $\sigma \subset I$, then $\sigma + (i,0,j,0) \in \mathcal{W}^\alpha$ for all $i,j \in \mathbf{Z}$, and if $\sigma \in \widetilde{\mathcal{W}}^\alpha$ is an n-simplex with $\sigma \subset I$, then $\sigma + (i,0,j,0) \in \widetilde{\mathcal{W}}^\alpha$ for all integers $-2 \le i,j \le 1$;
- (Congruent simplices away from the snowflake surface) The triangulations $\mathcal{W}^\alpha$, $1 \le \alpha < 2$, all contain a common subcollection $\mathcal{W}^*$ of n-simplices satisfying

$$\bigcup \{\sigma : \sigma \in \mathcal{W}^*\} = \mathbf{R}^4 \setminus J;$$

Extension. By Theorem 2.4, $\Gamma^1 \times \Gamma^1 = \mathbf{R}^2$ has the QSEP in $\mathbf{R}^4$. When $1 < \alpha < 2$, $\Gamma^\alpha \times \Gamma^\alpha$ is thick in $\mathbf{R}^4$. Since neither $\Gamma^\alpha \times \Gamma^\alpha$ nor its complement is bounded, Theorem 2.5 does not apply directly. However, the equivariance of the snowflake surfaces $\Gamma^\alpha \times \Gamma^\alpha$ and the corresponding canonical homeomorphisms $F_\alpha^{\alpha'}$ substitutes for the assumption of boundedness. It suffices to establish the following proposition.

Proposition 4.1. *For each $\alpha \in [1, 2)$ there exists $s_0 = s_0(\alpha) > 0$ so that every equivariant s-QS map $f : \Gamma^\alpha \times \Gamma^\alpha \to \mathbf{R}^4$ with $0 < s \le s_0$ has an equivariant s_1-QS extension $g : \mathbf{R}^4 \to \mathbf{R}^4$, where $s_1 = s_1(s, \alpha) \to 0$ as $s \to 0$. Thus g is K-QC, with $K = K(s_1) \to 1$ as $s_1 \to 0$ (see Theorem 2.3).*

Recall that the term "equivariant" refers to the specific group action in Remark 3.2.

In addition to the conclusion of Proposition 4.1, the process of constructing g and specific estimates which arise therein, play an essential role in our proof of Theorem 1.5. We emphasize that aside from the use of equivariance, the ideas in the following construction, and in particular, in the proof of Lemma 4.2, are due to Tukia and Väisälä. We follow closely the steps and notation from [27] and [31], and choose sets and constants that are, while not always the same, at least comparable to those from these references.

Fix $\alpha \in [1, 2)$. Following [27] and [31], choose an auxiliary parameter $q > 0$. To prove the extension property in Proposition 4.1, it suffices to find $q(\alpha) \in (0, 1)$ and, for each $q \in (0, q(\alpha)]$ a number $s = s(q, \alpha) > 0$ so that every equivariant s-QS map $f : \Gamma^\alpha \times \Gamma^\alpha \to \mathbf{R}^4$ has an equivariant s_1-QS extension $g : \mathbf{R}^4 \to \mathbf{R}^4$ with $s_1 = s_1(q, \alpha) \to 0$ as $q \to 0$. The function $s(q, \alpha)$ can be chosen to be increasing with respect to q, whence also $q = q(s, \alpha) \to 0$ as $s \to 0$.

We will prove the following lemma on extensions of the s-quasisymmetric map in Proposition 4.1. Here and henceforth we abbreviate

$$d^\alpha(x) := \operatorname{dist}(x, \Gamma^\alpha \times \Gamma^\alpha).$$

Lemma 4.2. *Let $0 < q < 1/10$, let $b = q^{-1/3}$ if $\alpha > 1$ and $b = 20$ if $\alpha = 1$, and assume that $s = s(q, \alpha)$ is sufficiently small. Then to each $x \in \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ there corresponds a similarity $h_x^\alpha : \mathbf{R}^4 \to \mathbf{R}^4$ so that the estimates*

$$||h_x^\alpha - h_y^\alpha||_{B(y, d^\alpha(y))} \le M(\alpha) q^{2/3} d^\alpha(x) L(h_x^\alpha) \tag{4.3}$$

and

$$L(h_y^\alpha) d^\alpha(y) \le M(\alpha) b L(h_x^\alpha) d^\alpha(x) \tag{4.4}$$

hold for all $x, y \in \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ satisfying $|y - x| < b d^\alpha(x)$. Moreover, there exists an extension g of f satisfying

$$||g - h_x^\alpha||_{B(x, b d^\alpha(x))} \le M(\alpha) q^{2/3} b d^\alpha(x) L(h_x^\alpha). \tag{4.5}$$

Here $M(\alpha) > 1$ denotes a constant depending only on α.

Compare [27, pp. 165–169] and [31, pp. 264–268].

Proof. First, consider the case $1 < \alpha < 2$. Let $0 < q < 1/10$ and $b = q^{-1/3}$. We consider only sufficiently small values of s so that

$$|| \operatorname{id} - f||_{A^\alpha} < q^2.$$

(In other words, we require $\varkappa(s,4) < q^2/12$ in Theorem 2.1.) Assign, to each $x \in \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$, a set $Q(x) := B(x, bd^\alpha(x)) \cap \Gamma^\alpha \times \Gamma^\alpha$ and a similarity $h_x^\alpha : \mathbf{R}^4 \to \mathbf{R}^4$ so that $h_x^\alpha = \operatorname{id}$ if $d^\alpha(x) \ge q$ and

$$||h_x^\alpha - f||_{Q(x)} \le \varkappa(s) L(h_x^\alpha) \operatorname{diam} Q(x)$$

if $d^\alpha(x) < q$. The existence of h_x^α follows from Theorem 2.1; we write $\varkappa(s)$ in place of $\varkappa(s,4)$.

Define an extension g of f as follows. At each vertex v of an n-simplex $\sigma \in \mathcal{W}^\alpha$, set

$$g(v) = h_v^\alpha(v),$$

extend g to $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ so that g is affine on each n-simplex in $\mathcal{W}^\alpha$, and let $g|_{\Gamma^\alpha \times \Gamma^\alpha} = f$. Observe that $g = \operatorname{id}$ on $\{d^\alpha > 2q\} \supset \mathbf{R}^4 \setminus J$, and that g is equivariant in the sense of Remark 3.2.

Let $\tilde f = f|_{A^\alpha}$. Extend $\tilde f$ from A^α to $\mathbf{R}^4$ by the same procedure. To each $x \in \mathbf{R}^4 \setminus A^\alpha$, assign a set $\tilde Q(x) = B(x, bd^\alpha(x)) \cap A^\alpha$ and a similarity $\tilde h_x^\alpha : \mathbf{R}^4 \to \mathbf{R}^4$ so that $\tilde h_x^\alpha = \operatorname{id}$ if $\operatorname{dist}(x, A^\alpha) \ge q$ and

$$||\tilde h_x^\alpha - f||_{\tilde Q(x)} \le \varkappa(s) L(\tilde h_x^\alpha) \operatorname{diam} \tilde Q(x)$$

if $\operatorname{dist}(x, A^\alpha) < q$. As in the previous paragraph, define $\tilde g(v) = \tilde h_v^\alpha(v)$ at each vertex v of an n-simplex $\sigma \in \widetilde{\mathcal{W}}^\alpha$, and extend $\tilde g$ to all of σ affinely. The equivariance of the map f and the simplices in $\mathcal{W}^\alpha$ and $\widetilde{\mathcal{W}}^\alpha$ together with the choice of q ensures that

$$\tilde g(x_1 + i, x_2, x_3 + j, x_4) = \tilde g(x_1, x_2, x_3, x_4) + (i, 0, j, 0) \tag{4.6}$$

whenever $0 \le x_1, x_3 \le 1$ and $i, j \in \{-2, -1, 0, 1\}$, and that

$$g(x_1, x_2, x_3, x_4) = \tilde g(x_1, x_2, x_3, x_4) \tag{4.7}$$

whenever $\max\{|x_1|, |x_3|\} \le 2$.

Because A^α is thick and bounded, it follows from the proof of Theorem 2.5 in [31] (see, in particular, the proofs of equations (6.3), (6.7) and (6.9) from [31]) that $s = s(q, \alpha)$ can be chosen small enough so that the estimates (4.3)–(4.5) hold for $x, y \in \mathbf{R}^4 \setminus A^\alpha$, with $\tilde h_x^\alpha$, $\tilde h_y^\alpha$, $\tilde g$ and $\operatorname{dist}(x, A^\alpha)$ in place of h_x^α, h_y^α, g and $d^\alpha(x)$. Then we obtain (4.3)–(4.5) as stated for $\alpha \in (1,2)$ by (4.6) and (4.7).

Now assume $\alpha = 1$. We follow the notation and constructions from section 5 of [27]. Assign, to each dyadic square $Q \subset \mathbf{R}^2$, a similarity $u_Q : \mathbf{R}^2 \to \mathbf{R}^4$ so that

$$||u_Q - f_Q||_Q \le \varkappa(s) L(u_Q) \operatorname{diam} Q;$$

the existence of $\varkappa(s)$ follows from Theorem 2.1. When $\{u_Q\}$ are chosen following certain additional rules, they can be extended to similarities $\{h_Q\}$ from $\mathbf{R}^4$ to $\mathbf{R}^4$ so that

$$||h_Q - h_{Q'}||_{Z_Q} \leq \epsilon(s) L(u_Q) \operatorname{diam} Q$$

for all dyadic squares Q and Q' satisfying $Q \cap Q' \neq \emptyset$ and $\frac{1}{2} \leq \operatorname{diam} Q / \operatorname{diam} Q' \leq 2$, where Z_Q is the cube in $\mathbf{R}^4$ concentric with Q of diameter $b \operatorname{diam} Q$ and edges parallel to the coordinate axes. (To avoid lengthy definition, the set Z_Q described in the previous sentence is not the same as that in [27, p. 168], however, they both contain Q in their center half and have comparable diameters.) The function $\epsilon(s)$ is derived from $\varkappa(s)$ and satisfies $\lim_{s \to 0} \epsilon(s) = 0$. Proper choice of the similarities $\{u_Q\}$ and their extensions $\{h_Q\}$ requires considerable work. Finally, to each $x \in \mathbf{R}^4 \setminus \mathbf{R}^2$, assign a dyadic square $Q(x) \subset \mathbf{R}^2$ following certain rules (as in pp. 159–163 of [27]). In particular, we require

$$1 \leq \frac{\operatorname{diam} Q(x)}{\operatorname{dist}(x, \mathbf{R}^2)} \leq 2$$

for all such x. Set $h^1_x = h_{Q(x)}$.

As before, define $g(v) = h^1_v(v)$ at each vertex v on an n-simplex $\sigma \in \mathcal{W}^1$, extend g to be affine on each such σ, and let $g = f$ on $\mathbf{R}^2$. Now it follows from the proof of Theorem 2.4 in [27] (see in particular equations (5.9) and (5.10)) that $s = s(q, 1)$ and hence also $\varkappa(s)$ and $\epsilon(s)$ can be chosen small enough so that (4.3) and (4.5) hold with $\alpha = 1$. The proof of the first inequality in [31, (6.7)] can be reproduced to give (4.4). □

Let us pause to discuss the dependence of the various constants that have arisen. Recall that the constants C_i, $1 \leq i \leq 5$, from Remark 2.6, (2.8), (2.9), (2.11), (2.14) and (2.15) depend only on the dimension $n = 4$, and hence are absolute constants.

When $1 < \alpha < 2$, $M(\alpha)$ depends on $n = 4$, on the diameter of A^α, and on the numbers $r_0 = 1$ and $\beta(\alpha)$ describing the thickness of $\Gamma^\alpha \times \Gamma^\alpha$. Thus $M(\alpha)$ depends only on α, and is nonincreasing. In fact $M(\alpha) \to \infty$ as $\alpha \to 1$.

The constant $M(1)$ is derived from a different argument (Theorem 2.4) where thickness is not used. In fact, $M(1)$ depends on the dimension $n = 4$ only, and hence is an absolute constant.

Continuing with the proof of Proposition 4.1, choose

$$q(\alpha) = (10^{10} C_1 C_4 C_5 M(\alpha))^{-6}. \tag{4.8}$$

We assume from now on that $0 < q < q(\alpha)$, so that

$$10^{10} C_1 C_4 C_5 M(\alpha) q^{1/3} < q^{1/6}. \tag{4.9}$$

A further restriction on q will be imposed during the proof of Proposition 5.5 in order to guarantee that Theorem 5.3 is applicable.

Lemma 4.10. *The map g defined in Lemma 4.2 is a sense preserving quasiconformal map of $\mathbf{R}^4$, whose metric dilatation $H(g)$ satisfies*

$$H(g) \le (1 + 2M(\alpha)q^{1/3})^2 \tag{4.11}$$

a.e. in $\mathbf{R}^4$.

For the definition of the metric dilatation, see [29].

We shall prove Lemma 4.10 for g (when $\alpha = 1$) and for $\tilde{g}$ (when $1 < \alpha < 2$) following the ideas in the proofs of Theorems 2.4 and 2.5 respectively, together with Theorem 2.3. Lemma 4.10 then holds for the map g (when $1 < \alpha < 2$) by the equivariance, and the fact that g and $\tilde{g}$ coincide on $\{x : |x_1| \le 2 \text{ and } |x_3| \le 2\}$.

As in the proofs of Theorems 2.4 and 2.5, the continuity of g is straightforward, and the sense preservation, injectivity and surjectivity can be deduced from the estimates in Lemma 4.2 and topological considerations.

On each n-simplex $\sigma \in \mathcal{W}^\alpha$, g is affine. It follows from Lemma 4.2 and Lemma 4.12 below that the metric dilatation $H(g)(x)$ is bounded by $(1+2M(\alpha)q^{1/3})^2$, uniformly on the union of the interiors of the n-simplices in $\mathcal{W}^\alpha$. Observing that the set $\cup\{\partial\sigma : \sigma \in \mathcal{W}^\alpha\}$ has σ-finite 3-dimensional measure, it follows from a removability theorem (Theorem 35.1 of [29]) that g is quasiconformal in $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$. The verification of the boundedness of $H(g)$ requires some work; it has been given by Väisälä in [31, p. 260]. Finally, since $\Gamma^\alpha \times \Gamma^\alpha$ has 4-measure zero, g is quasiconformal on all of $\mathbf{R}^4$, see [29]. Moreover, g is s_1-quasisymmetric with $s_1 = s_1(q, \alpha) \to 0$ as $q \to 0$.

In the preceding paragraph the following lemma was used.

Lemma 4.12. *Let h be a similarity and let g be an affine map. If*

$$||g - h||_{B(x,r)} < \lambda r L(h),$$

for some $0 < \lambda < 1/100$, then

$$|g' - h'| \le \lambda L(h)$$

and

$$H(g) \le (1 + 2\lambda)^2,$$

moreover, g is sense-preserving if and only if h is sense-preserving.

Here we denote by $|A|$ the operator norm of a matrix A.

Proof. Note that $|g(x+y) - h(x+y)| < \lambda r L(h)$ when $|y| < r$. Replacing y by $-y$ and subtracting, we obtain by linearity the estimate $2|g'(y) - h'(y)| \le 2\lambda r L(h)$ when $|y| < r$. Thus $|g' - h'| \le \lambda L(h)$. The remaining statements can be found in [27, 3.5] and [31, 2.7]. □

Estimates. We now derive from Lemma 4.2 a few estimates for the extension g; some of these have been implicitly used in [27] and [31].

Lemma 4.13. *For almost every x and y in $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$, the following estimates hold:*

(i) $|(h_x^\alpha)' - (h_y^\alpha)'| \le 2M(\alpha)q^{2/3}L(h_x^\alpha)$ *and*

$$1 - 2M(\alpha)q^{2/3} \le L(h_y^\alpha)/L(h_x^\alpha) \le 1 + 2M(\alpha)q^{2/3}$$

if $|y-x| \le d^\alpha(x)/2$*;*

(ii) $|g'(x) - (h_x^\alpha)'| \le M(\alpha)q^{2/3}L(h_x^\alpha)$ *and*

$$1 - M(\alpha)q^{2/3} \le \frac{|g'(x)|}{L(h_x^\alpha)} \le 1 + M(\alpha)q^{2/3};$$

(iii) $|g'(x) - g'(y)| \le 5M(\alpha)q^{2/3}L(h_x^\alpha)$ *if* $|y-x| \le d^\alpha(x)/2$*;*

(iv)

$$1 - 4M(\alpha)q^{2/3} \le \frac{|g(x)-g(y)|}{|x-y|L(h_x^\alpha)} \le 1 + 4M(\alpha)q^{2/3}$$

if $|y-x| \le d^\alpha(x)/2$*, and*

$$|g(x)-g(y)| \le (|x-y| + 2M(\alpha)q^{2/3}bd^\alpha(x))L(h_x^\alpha)$$

if $|y-x| \le bd^\alpha(x)$*;*

(v)

$$(\frac{1}{2} - 3M(\alpha)q^{2/3}) \le \frac{\operatorname{dist}(g(x), f(\Gamma^\alpha \times \Gamma^\alpha))}{d^\alpha(x)L(h_x^\alpha)} \le (1 + 2M(\alpha)b\,q^{2/3}).$$

Proof. Recall that $b = q^{-1/3}$ when $\alpha > 1$ and $b = 20$ when $\alpha = 1$. Thus (i) and (ii) follow from Lemmas 4.2 and 4.12, (iii) follows from (i), (ii) and the triangle inequality, and (iv) follows from Lemma 4.2, (i), (ii) and the triangle inequality.

Let z be a point in $\Gamma^\alpha \times \Gamma^\alpha$ with $|x-z| = d^\alpha(x)$. Then

$$|g(x) - f(z)| = |g(x) - g(z)| \le (1 + 2M(\alpha)bq^{2/3})d^\alpha(x)L(h_x^\alpha)$$

by (iv); this proves the right hand inequality in (v). On the other hand, for any w with $|w-x| = d^\alpha(x)/2$,

$$|g(x) - g(w)| \ge (\frac{1}{2} - 3M(\alpha)q^{2/3})d^\alpha(x)L(h_x^\alpha)$$

by (ii) and (iii); this proves the left hand inequality in (v). □

5. Smoothing

For $\alpha \in [1,2)$, let $A = \Gamma^\alpha \times \Gamma^\alpha$ and choose and fix a regularized distance function $\delta^\alpha = \delta_A$ to A which satisfies properties (2.7)–(2.9). Define

$$G(x) = \begin{cases} \delta^\alpha(x)^{-4} \int_{\mathbf{R}^4} g(y)\varphi(\frac{x-y}{\delta^\alpha(x)})\,dy, & \text{on } \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha, \\ g(x), & \text{on } \Gamma^\alpha \times \Gamma^\alpha, \end{cases}$$

where g is the extension in the previous section.

Lemma 5.1. *For almost every x and y in $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$, the following estimates hold:*

(i) $|G'(x) - g'(x)| \le 12 \cdot 10^4 C_1 C_4 M(\alpha) q^{1/3} L(h_x^\alpha)$;

(ii) $|G(x) - g(x)| \le 3 \cdot 10^4 C_1 C_4 M(\alpha) q^{1/3} d(x) L(h_x^\alpha)$;

(iii)

$$\frac{1}{2} - 10^5 C_1 C_4 M(\alpha) q^{1/3} \le \frac{\operatorname{dist}(G(x), f(\Gamma^\alpha \times \Gamma^\alpha))}{d^\alpha(x) L(h_x^\alpha)} \le 1 + 10^5 C_1 C_4 M(\alpha) q^{1/3}$$

and

$$1 - 10^6 C_1 C_4 M(\alpha) q^{1/3} \le \frac{|G'(x)|}{L(h_x^\alpha)} \le 1 + 10^6 C_1 C_4 M(\alpha) q^{1/3};$$

(iv)

$$|G'(x) - G'(y)| \le 10^9 C_1^2 C_5 M(\alpha) |x - y| L(h_x^\alpha)/d^\alpha(x)$$

and

$$|G'(x)^{-1} - G'(y)^{-1}| \le 4 \cdot 10^9 C_1^2 C_5 M(\alpha) |x - y| / (L(h_x^\alpha) d^\alpha(x))$$

if $y \in B(x, d^\alpha(x)/2)$;

(v)

$$1 - 10^5 C_1 C_4 M(\alpha) q^{1/3} \le \frac{|G(x) - G(y)|}{|x - y| L(h_x^\alpha)} \le 1 + 10^5 C_1 C_4 M(\alpha) q^{1/3}$$

if $y \in B(x, d^\alpha(x)/2)$.

Proof. Since α is fixed, we write d, δ, h_x for d^α, δ^α and h_x^α. Suppose that x is in the interior of some n-simplex σ in $\mathcal{W}^\alpha$. Then $g^\sigma := g|_\sigma$ is an affine map, and we write $g^\sigma(z) = Bz + b$. By Lemma 2.12,

$$\begin{aligned} G(x) - g(x) &= G(x) - Bx - b \\ &= \delta^{-4}(x) \int_{\mathbf{R}^4} (g(y) - By - b) \varphi\left(\frac{x - y}{\delta(x)}\right) dy. \end{aligned}$$

Then by Proposition 2.13,

$$\left| \frac{\partial}{\partial x_j}(G_i - g_i)(x) \right| \le \frac{C_4}{\delta(x)} \operatorname{Osc}(g - B - b, x, \delta(x))$$

for all $i, j = 1, \ldots, 4$. Note that $g(x) = g^\sigma(x) = Bx + b$, so

$$\begin{aligned} |(g(y) - By - b) - (g(x) - Bx - b)| &= |g(y) - g^\sigma(y)| \\ &\le |g(y) - h_x(y)| + |h_x(y) - g^\sigma(y)| \end{aligned}$$

for $y \in B(x, \delta(x))$. We get $|g(y) - h_x(y)| \le M(\alpha) q^{1/3} d(x) L(h_x^\alpha)$ from Lemma 4.2, and

$$\begin{aligned} |h_x(y) - g^\sigma(y)| &\le |h_x(y) - g^\sigma(y) - h_x(x) + g^\sigma(x)| + |h_x(x) - g^\sigma(x)| \\ &\le |h_x' - g'(x)| \cdot |y - x| + |h_x(x) - g(x)| \\ &\le M(\alpha) q^{2/3} \delta(x) L(h_x) + M(\alpha) q^{1/3} d(x) L(h_x) \\ &\le 2M(\alpha) q^{1/3} d(x) L(h_x) \end{aligned}$$

by Lemma 4.2 and Lemma 4.13(ii). Since $d(x) < 10^4 C_1 \delta(x)$,

$$\left| \frac{\partial}{\partial x_j}(G_i - g_i)(x) \right| \le 3 \cdot 10^4 C_1 C_4 M(\alpha) q^{1/3} L(h_x).$$

This proves (i).

Again assume $x \in \sigma$ for some n-simplex $\sigma \in \mathcal{W}^\alpha$. By Remark 2.10, σ contains a ball $B(x_\sigma, 2\delta(x))$. Since g is affine on σ, $G(x_\sigma) = g(x_\sigma)$ (Lemma 2.12), and $|x - x_\sigma| \le \operatorname{diam} \sigma \le d(x)/4$, we have

$$|G(x) - g(x)| = |G(x) - g(x) - (G(x_\sigma) - g(x_\sigma))| \le 3 \cdot 10^4 C_1 C_4 M(\alpha) q^{1/3} L(h_x) d(x).$$

This proves (ii).

The first part of (iii) follows from (ii) and Lemma 4.13(v). The second part follows from (i) and Lemma 4.13(ii):

$$|G'(x) - h'_x| \le |G'(x) - g'(x)| + |g'(x) - h'_x| \le 10^6 C_1 C_4 M(\alpha) q^{1/3}.$$

To prove (iv), we use the second derivative estimates for the convolution in Proposition 2.13 together with the second part of Lemma 4.13(iv). We get

(5.2)

$$\begin{aligned} \left| \frac{\partial}{\partial x_j} G_i(x) - \frac{\partial}{\partial x_j} G_i(y) \right| &\le \frac{C_5}{\delta^2(x)} \operatorname{Osc}(g, x, 3d(x)/4) |x - y| \\ &\le \frac{C_5}{\delta^2(x)} \left(\frac{3}{2} d(x) + 2M(\alpha) b q^{2/3} d(x) \right) L(h_x) |x - y| \\ &\le 2 \cdot 10^8 C_1^2 C_5 |x - y| L(h_x)/d(x), \end{aligned}$$

which gives the first inequality in (iv).

The second inequality follows from the first together with (iii) and (4.9):

$$\begin{aligned} |G'(x)^{-1} - G'(y)^{-1}| &= |G'(x)^{-1}(G'(y) - G'(x)) G'(y)^{-1}| \\ &\le 4 \cdot 10^9 C_1^2 C_5 |x - y| / (L(h_x) d(x)). \end{aligned}$$

Finally, (v) follows from (i) and Lemma 4.13(i),(ii). □

To verify the quasiconformality of G, we use some results from the general theory of quasiregular maps. Theorem 5.3 is due to Gol'dshteĭn and Theorem 5.4 is due to Zorich. See [20, Theorem VI.8.14 and Corollary III.3.8].

Theorem 5.3. *For each $n \ge 3$ there exists a constant $K_0 > 1$ so that every nonconstant K_0-quasiregular map $f : U \to \mathbf{R}^n$, U a domain in $\mathbf{R}^n$, is a local homeomorphism.*

Theorem 5.4. *Each locally homeomorphic quasiregular map $f : \mathbf{R}^n \to \mathbf{R}^n$, $n \ge 3$, is a homeomorphism, hence quasiconformal.*

Proposition 5.5. *G is a quasiconformal homeomorphism on $\mathbf{R}^4$, and G and G^{-1} are C^∞ on $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ and $\mathbf{R}^4 \setminus f(\Gamma^\alpha \times \Gamma^\alpha)$, respectively.*

Proof. Continuity on $\mathbf{R}^4$ and C^∞-smoothness on $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ for G follow from Proposition 2.13.

By Lemma 5.1(iii) and Lemma 4.13(ii), the ratio $|G'(x)|/|g'(x)|$ is uniformly bounded away from zero and infinity on the complement of $\Gamma^\alpha \times \Gamma^\alpha$. Since almost every line segment parallel to one of the coordinate axes meets $\Gamma^\alpha \times \Gamma^\alpha$ in a set of length zero and g is ACL, G is also ACL. Moreover, since $\Gamma^\alpha \times \Gamma^\alpha$ has 4-measure zero and g is ACL^4, G is also ACL^4.

By Lemma 5.1(v), the metric dilatation of G on $\mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$ is bounded by $(1+10^6 C_1 C_4 M(\alpha) q^{1/3})^2$. Suppose now that $y \in \Gamma^\alpha \times \Gamma^\alpha$ and $x \notin \Gamma^\alpha \times \Gamma^\alpha$; then

$$\begin{aligned} |G(y) - G(x)| &= |g(y) - G(x)| \\ &\geq |g(y) - g(z)| + |g(z) - g(x)| - |g(x) - G(x)| \end{aligned}$$

where z is the point on $\partial B(x, d(x)/2)$ with $g(z)$ on the line segment $\overline{g(x)g(y)}$. By Lemma 5.1(ii) and Lemma 4.13(iv),

$$\begin{aligned} |g(x) - G(x)| &< 3 \cdot 10^4 C_1 C_4 M(\alpha) q^{1/3} d(x) L(h_x^\alpha) \\ &< 10^6 C_1 C_4 M(\alpha) q^{1/3} |g(z) - g(x)|; \end{aligned}$$

whence

$$|G(y) - G(x)| > (1 - 10^6 C_1 C_4 M(\alpha) q^{1/3}) |g(y) - g(x)|.$$

On the other hand,

$$\begin{aligned} |G(y) - G(x)| &\leq |g(y) - g(z)| + |g(z) - g(x)| + |g(x) - G(x)| \\ &< (1 + 10^6 C_1 C_4 M(\alpha) q^{1/3}) |g(y) - g(x)|. \end{aligned}$$

Since $H(g)$ is uniformly bounded on $\mathbf{R}^4$ (see (4.11)), we conclude from the above estimates together with the continuity of G that $H(G)$ is bounded on $\mathbf{R}^4$ by

$$(1 + 10^7 C_1 C_4 M(\alpha) q^{1/3})^2 H(g) \leq (1 + 10^7 C_1 C_4 M(\alpha) q^{1/3})^4.$$

At each point $x \in \mathbf{R}^4 \setminus \Gamma^\alpha \times \Gamma^\alpha$, G is differentiable and we have

$$\frac{|G'(x)|^4}{\det G'(x)} \leq H(G)^3 \leq (1 + 10^7 C_1 C_4 M(\alpha) q^{1/3})^{12}.$$

Thus G is K-QR with

$$K := (1 + 10^7 C_1 C_4 M(\alpha) q^{1/3})^{12}.$$

For sufficiently small q we have $K \leq K_0$, where K_0 is the constant from the statement of Theorem 5.3. It follows from Theorem 5.3 that G is a local homeomorphism, and then from Theorem 5.4 that G is a homeomorphism. Thus G is quasiconformal.

The C^∞-smoothness of G^{-1} on $\mathbf{R}^4 \setminus f(\Gamma^\alpha \times \Gamma^\alpha)$ follows from the injectivity of G.

Finally, surjectivity of G follows from the quasiconformality. □

6. Proof of Theorem 1.5

Given $\epsilon > 0$, let $\alpha = 2 - \epsilon$. Applying Proposition 3.3(d) with s_0 chosen as in Proposition 4.1, we find a finite sequence

$$1 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = \alpha$$

so that the canonical map $f_k := F_{\alpha_k}^{\alpha_{k+1}}$ (in the notation of section 3) from $\Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ to $\Gamma^{\alpha_{k+1}} \times \Gamma^{\alpha_{k+1}}$ admits a quasiconformal extension g_k to $\mathbf{R}^4$. Let G_k be a smoothing of g_k on $\mathbf{R}^4 \setminus \Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$, constructed as in section 5. Let $H_k = G_k^{-1}$,

$$G = G_{m-1} \circ G_{m-2} \circ \cdots \circ G_0$$

and

$$H = G^{-1} = H_0 \circ \cdots \circ H_{m-2} \circ H_{m-1}.$$

Let $U = \mathbf{R} \times (-2, 2) \times \mathbf{R} \times (-2, 2)$. From the extension construction in section 4 and the smoothing process in section 5, it follows that $G_k = \mathrm{id}$ on $\mathbf{R}^4 \setminus U$ for all k, whence also $H_k, G, H = \mathrm{id}$ on $\mathbf{R}^4 \setminus U$.

Proposition 6.1. *Assume $X, Y \in U$. There exists $\lambda(\alpha) > 1$ so that*

$$\lambda(\alpha)^{-1} d(X, \Gamma^\alpha \times \Gamma^\alpha)^{\alpha-1} \le |H'(X)| \le \lambda(\alpha) d(X, \Gamma^\alpha \times \Gamma^\alpha)^{\alpha-1}, \tag{6.2}$$

$$\lambda(\alpha)^{-1}|X-Y|^\alpha \le |H(X)-H(Y)| \le \lambda(\alpha)|X-Y|^\alpha, \qquad \forall X \in \Gamma^\alpha \times \Gamma^\alpha, |X-Y| \le 1, \tag{6.3}$$

and

$$|H'(X) - H'(Y)| \le \lambda(\alpha)|X-Y|^{\alpha-1}, \qquad \forall |X-Y| \le 1. \tag{6.4}$$

The map H is $C^{2-\epsilon}$-smooth, quasiconformal on $\mathbf{R}^4$, and maps $\Gamma^\alpha \times \Gamma^\alpha$ onto $\mathbf{R}^2$.

Let $d \ge 2$ be an integer, and let $w : \mathbf{R}^4 \to \mathbf{R}^4$ be the winding map

$$w(x_1, x_2, x_3, x_4) = (x_1, r \cos d\theta, x_3, r \sin d\theta),$$

where (r, θ) denote polar coordinates in the x_2x_4-plane. We observe the following properties of w:

- w is quasiregular with branch set $\mathbf{R}^2 = \{(x_1, 0, x_3, 0) : x_1, x_3 \in \mathbf{R}\}$ [20, p. 13].
- w is Lipschitz of order one on $\mathbf{R}^4$.
- $w = (w_1, w_2, w_3, w_4)$ is C^∞-smooth and

$$\max_{k,i,j} \left| \frac{\partial^2 w_k}{\partial x_i \partial x_j}(x) \right| \simeq (x_2^2 + x_4^2)^{-1/2}$$

  on $\mathbf{R}^4 \setminus \mathbf{R}^2$.

The composition $F := w \circ H$ is a degree d quasiregular map on $\mathbf{R}^4$ whose branch set $\Gamma^\alpha \times \Gamma^\alpha$ has Hausdorff dimension $4 - 2\epsilon$. The $C^{2-\epsilon}$-smoothness of F can be established from properties of w and Proposition 6.1.

Remark 6.5. The snowflake property of H in Proposition 6.1 is essential in establishing the $C^{2-\epsilon}$-smoothness of F, since w is only Lipschitz continuous.

It remains to prove Proposition 6.1. We introduce the abbreviated notations $d_k = d^{\alpha_k}$ and $h_{k,x} = h_x^{\alpha_k}$.

Proposition 6.6. *Consider $x, y \in U$.*

(i) $|G_k(x)-G_k(y)| \simeq |x-y|^{\alpha_k/\alpha_{k+1}}$ *for all* $x \in \Gamma^{\alpha_k}\times\Gamma^{\alpha_k}$ *and all* $|x-y| \le 1$, *and* $|H_k(X)-H_k(Y)| \simeq |X-Y|^{\alpha_{k+1}/\alpha_k}$ *for all* $X \in \Gamma^{\alpha_{k+1}}\times\Gamma^{\alpha_{k+1}}$ *and all* $|X-Y| \le 1$;

(ii) $d_{k+1}(G_k(x)) \simeq d_k(x)^{\alpha_k/\alpha_{k+1}}$ *and*

$$|G_k'(x)| \simeq L(h_{k,x}) \simeq d_k(x)^{(\alpha_k/\alpha_{k+1})-1}$$

for all $x \in U \setminus \Gamma^{\alpha_k}\times\Gamma^{\alpha_k}$;

(iii) H_k' *exists on all of* U *and* $H_k'(X) = 0$ *for* $X \in \Gamma^{\alpha_{k+1}}\times\Gamma^{\alpha_{k+1}}$;

(iv) $|H_k'(X) - H_k'(Y)| \preceq |X-Y|^{(\alpha_{k+1}/\alpha_k)-1}$ *for all* $|X-Y| \le 1$.

Here the notation $A \preceq B$, respectively $A \simeq B$, means that there exists a constant C depending only on ϵ so that $A \le CB$, respectively $C^{-1}B \le A \le CB$.

Proof. To prove (i), we recall from Proposition 3.3 and the fact that $G_k = g_k = f_k$ on $\Gamma^{\alpha_k}\times\Gamma^{\alpha_k}$ that $|G_k(x)-G_k(y)| \simeq |x-y|^{\alpha_k/\alpha_{k+1}}$ for all $x, y \in \Gamma^{\alpha_k}\times\Gamma^{\alpha_k}$, $|x-y| \le 1$. If $x \in \Gamma^{\alpha_k}\times\Gamma^{\alpha_k}$ and $y \notin \Gamma^{\alpha_k}\times\Gamma^{\alpha_k}$, choose $z \in \Gamma^{\alpha_k}\times\Gamma^{\alpha_k}$ with $|x-z| = |x-y|$; since the metric dilatations $H(G_k)$ are uniformly bounded on $\mathbf{R}^4$, we have

$$|G_k(x)-G_k(y)| \simeq |G_k(x)-G_k(z)| \simeq |x-z|^{\alpha_k/\alpha_{k+1}} = |x-y|^{\alpha_k/\alpha_{k+1}}.$$

The estimates on H_k follow by taking the inverse.

To prove (ii), let $X = G_k(x)$, choose $Y \in \Gamma^{\alpha_{k+1}}\times\Gamma^{\alpha_{k+1}}$ so that $|X-Y| = d_{k+1}(X)$, and let $y = G_k^{-1}(Y)$. Since $x \in U$, we have $X \in U$ and $|X-Y| \preceq 1$. Then

$$d_k(x) \le |x-y| \simeq |X-Y|^{\alpha_{k+1}/\alpha_k} = d_{k+1}(X)^{\alpha_{k+1}/\alpha_k}$$

by (i). For the inverse, choose $z \in \Gamma^{\alpha_k}\times\Gamma^{\alpha_k}$ satisfying $|x-z| = d_k(x)$, and let $Z = G_k(z)$; then $|x-z| \preceq 1$ and

$$d_{k+1}(X) \le |X-Z| \simeq |x-z|^{\alpha_k/\alpha_{k+1}} = d_k(x)^{\alpha_k/\alpha_{k+1}}.$$

This proves the first part of (ii). The second half of (ii) follows from Lemma 5.1(iii).

Part (iii) is a direct consequence of (i) and Proposition 5.5.

To prove (iv), let $x = G_k^{-1}(X)$ and $y = G_k^{-1}(Y)$. We consider three cases; any remaining cases are covered by interchanging x and y. Assume that $|X-Y| \le 1$.

Case 1. $y \in \Gamma^{\alpha_k}\times\Gamma^{\alpha_k}$. By Lemma 5.1(iii) and parts (ii) and (iii) of this proposition,

$$\begin{aligned}|H_k'(X)-H_k'(Y)| &= |H_k'(X)| = |G_k'(x)^{-1}| \\ &\simeq d_{k+1}(X)^{(\alpha_{k+1}/\alpha_k)-1} \le |X-Y|^{(\alpha_{k+1}/\alpha_k)-1}.\end{aligned}$$

Assume, from now on, that neither x nor y is on $\Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$.

Case 2. $y \in B(x, d_k(x)/2)$. By Lemma 5.1(iv),(v) and part (ii) of this proposition,

$$|H_k'(X) - H_k'(Y)| \preceq \frac{|x-y|}{d_k(x)} L(h_{k,x})^{-1} \preceq 2\frac{|X-Y|}{d_k(x)} L(h_{k,x})^{-2}$$
$$\simeq |X-Y| d_{k+1}(X)^{(\alpha_{k+1}/\alpha_k)-2} \preceq |X-Y|^{(\alpha_{k+1}/\alpha_k)-1}.$$

Case 3. $y \notin B(x, d_k(x)/2)$ and $x \notin B(y, d_k(y)/2)$. We assume, as we may, that $d_k(x) \le d_k(y) \le 2|x-y|$. Choose z on $\Gamma^{\alpha_k} \times \Gamma^{\alpha_k}$ so that $|x-z| \simeq d_k(x)$ and $|y-z| \simeq |x-y|$, and let $Z = G_k(z)$. Then

$$\begin{aligned}|H_k'(X) - H_k'(Y)| &\le |H_k'(X) - H_k'(Z)| + |H_k'(Z) - H_k'(Y)| \\ &\preceq |X-Z|^{(\alpha_{k+1}/\alpha_k)-1} + |Z-Y|^{(\alpha_{k+1}/\alpha_k)-1} \\ &\preceq |X-Y|^{(\alpha_{k+1}/\alpha_k)-1},\end{aligned}$$

by applying Case 1 twice and using the fact that the maps (G_k) are K-quasiconformal in $\mathbf{R}^4$ for a common K.

This completes the proof of Proposition 6.6. □

Lemma 6.7. *There exists $a(\alpha) > 1$ so that whenever $|x-y| \le 1$ and $0 \le k \le m$, then*

$$|G_{k+l} \circ \cdots \circ G_k(x) - G_{k+l} \circ \cdots \circ G_k(y)| \le a(\alpha)$$

for $l = 0, 1, \ldots, m-1-k$ and

$$|H_{k-j} \circ \cdots \circ H_{k-1}(x) - H_{k-j} \circ \cdots \circ H_{k-1}(y)| \le a(\alpha)$$

for $j = 1, 2, \ldots, k$.

The proofs of (6.2), (6.3) and (6.4) follow from Proposition 6.6, Lemma 6.7, and the chain rule. This completes the proof of Proposition 6.1 and hence also completes the proof of Theorem 1.5.

7. Proof of Theorem 1.6

The proof of Theorem 1.6 relies heavily on the construction by David and Toro [7] of codimension one snowflake surfaces. See [7], specifically Theorem 2.10 with $Z = R^{n-2}$ and $f(r) = \max\{1, r^{-\epsilon/(1+\epsilon)}\}$ and the discussion following (13.33).

Theorem 7.1 (David–Toro). *For each $n \ge 3$ there exists $\epsilon_0(n) > 0$ and $C = C(n) > 1$ so that for each $\epsilon \in (0, \epsilon_0(n))$ there exists a K-quasiconformal map $\Phi : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ with*

$$C^{-1}|x-y|^{1/(1+\epsilon)} \le |\Phi(x) - \Phi(y)| \le C|x-y|^{1/(1+\epsilon)} \tag{7.2}$$

for all $x, y \in \mathbf{R}^{n-2}$, $|x-y| \le 1$. Furthermore, $K \to 1$ as $\epsilon \to 0$.

David and Toro prove significantly stronger results; the source space $\mathbf{R}^{n-2}$ may be replaced by a metric space (Z,d) satisfying a Reifenberg flatness condition and the snowflaking behavior in (7.2) may be replaced by Orlicz-type conditions

$$C^{-1}|x-y|f(|x-y|) \le |\Phi(x)-\Phi(y)| \le C|x-y|f(|x-y|)$$

for a variety of gauge functions $f(r)$.

By the celebrated extension theorem of Tukia–Väisälä [26], the map Φ in Theorem 7.1 may be further extended to a quasiconformal map of $\mathbf{R}^n$. We continue to denote this extension by Φ. Observe that the extension procedure in [26] is ostensibly different from that in [27] and [31] used in section 4. It is therefore not obvious whether the smoothing procedure developed in section 5 can be applied directly to the extended map Φ. We bypass the issue by an alternative argument. Choose ϵ sufficiently small so that the David–Toro map Φ in Theorem 7.1 is K-quasiconformal with K very close to one, hence s-quasisymmetric with s very close to zero (see Theorem 2.3). Then $\varphi = \Phi|_{\mathbf{R}^{n-2}}$ may be re-extended to a quasiconformal map $g : \mathbf{R}^n \to \mathbf{R}^n$ by the Tukia–Väisälä extension procedure from [27] and [31] (Theorem 2.4). The smoothing procedure from section 5 applies to g, yielding a quasiconformal map G on $\mathbf{R}^n$ whose inverse has the snowflake property in Proposition 6.1. The desired quasiregular map in Theorem 1.5 is obtained as the composition of a winding map with G^{-1}.

Proof of Theorem 1.6. Let $n \ge 5$ and $d \ge 2$ be given. According to Theorem 2.4, $\mathbf{R}^{n-2}$ has the quasisymmetric extension property in $\mathbf{R}^n$; choose $s_0 > 0$ so that every s-quasisymmetric embedding $\varphi : \mathbf{R}^{n-2} \to \mathbf{R}^n$ with $s < s_0$ admits a quasisymmetric extension. Next, choose $K > 1$ so that every K-quasiconformal map of $\mathbf{R}^n$ is s-quasisymmetric for some $s \in (0, s_0)$ (Theorem 2.3), and choose $\epsilon > 0$ so that the map Φ from Theorem 7.1 is K-quasiconformal. Let $\varphi = \Phi|_{\mathbf{R}^{n-2}}$ and apply Theorem 2.4 (with $p = n-2$) to extend φ to an s_1-quasisymmetric homeomorphism $g : \mathbf{R}^n \to \mathbf{R}^n$ following the procedure summarized in section 4 in the case $\alpha = 1$. The smoothing procedure in section 5 applies to g and yields a quasiconformal map $G : \mathbf{R}^n \to \mathbf{R}^n$, C^∞-smooth on $\mathbf{R}^n \setminus \mathbf{R}^{n-2}$, with $G = \varphi$ on $\mathbf{R}^{n-2}$. As in Proposition 6.1, the snowflake property

$$C^{-1}|X-Y|^{1+\epsilon} \le |G^{-1}(X) - G^{-1}(Y)| \le C|X-Y|^{1+\epsilon} \tag{7.3}$$

holds for all $X \in \Sigma = \varphi(\mathbf{R}^{n-2})$ and all $Y \in \mathbf{R}^n$ with $|X-Y| \le 1$. Here C denotes a suitable constant depending only on the dimension n.

Let $w : \mathbf{R}^n \to \mathbf{R}^n$ be the degree d winding map

$$w(x_1, \ldots, x_{n-2}, r\cos\theta, r\sin\theta) = (x_1, \ldots, x_{n-2}, r\cos d\theta, r\sin d\theta),$$

where (r,θ) are polar coordinates in the $x_{n-1}x_n$-plane. The mapping w is quasiregular with branch set $\mathbf{R}^{n-2}$. Using (7.3) together with properties of w, it follows that $F = w \circ G^{-1}$ is a $C^{1+\epsilon}$-smooth degree d quasiregular map on $\mathbf{R}^n$ with branch set $B_F = \Sigma$. The proof of Theorem 1.6 is complete. □

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