BI-LIPSCHITZ EMBEDDINGS OF COMPACTA HYPERSPACES

JEREMY T. TYSON

Abstract. We study the bi-Lipschitz embedding problem for metric compacta hyperspaces. We observe that the compacta hyperspace $K(X)$ of any separable, uniformly disconnected metric space $X$ admits a bi-Lipschitz embedding in $\ell^2$. If $X$ is a countable compact metric space containing at most $n$ non-isolated points, there is a Lipschitz embedding of $K(X)$ in $\mathbb{R}^{n+1}$; in the presence of an additional convergence condition, this embedding may be chosen to be bi-Lipschitz. By way of contrast, the hyperspace $K([0, 1])$ of the unit interval contains a bi-Lipschitz copy of a certain self-similar doubling series-parallel graph studied by Laakso, Lang–Plaut, and Lee–Mendel–Naor, and consequently admits no bi-Lipschitz embedding into any uniformly convex Banach space. Schori and West proved that $K([0, 1])$ is homeomorphic with the Hilbert cube, while Hohti showed that $K([0, 1])$ is not bi-Lipschitz equivalent with a variety of metric Hilbert cubes.
1. Introduction

The topological structure of hyperspaces has been intensively studied during the twentieth century. According to a celebrated result of Curtis and Schori [7], the hyperspace of any nondegenerate Peano continuum is homeomorphic with the Hilbert cube. An essential reference for the topology of hyperspaces is the book by Illanes and Nadler [22].

Hyperspaces of metric spaces are naturally metrized by the Hausdorff metric. They play a role in dynamical systems, as the existence of attractors for iterated function systems on complete metric spaces may be demonstrated by an application of a suitable fixed point theorem to the hyperspace. This approach to fractal geometry was first presented by Hutchinson [21] (but see Moran [31] for earlier ideas along the same lines) and is now standard.

Motivated by these and other applications, some recent studies have focused on the intrinsic metric geometry of hyperspaces. Boardman, Goodey and McClure [13], [14], [5], [30] studied the generalized Hausdorff measures of hyperspaces of self-similar sets, while Hohti [20] showed that the hyperspace of the closed unit interval is not bi-Lipschitz equivalent with a variety of metric Hilbert cubes. Gruber, Lettl and Tichy [15], [16], [17] investigated the isometries of Euclidean and general hyperspaces. See also Bandt [3].

In this paper we study the existence of bi-Lipschitz embeddings of compacta hyperspaces in specified target spaces. First, we consider the case of Hilbert or finite-dimensional Euclidean targets. We observe that the hyperspace of any separable, uniformly disconnected metric space admits a bi-Lipschitz embedding in $\ell^2$. On the other hand, we prove that the hyperspace of any countable compact metric space containing at most $n$ non-isolated points admits a Lipschitz embedding in $\mathbb{R}^{n+1}$; provided a further convergence criterion is satisfied, the embedding in question is bi-Lipschitz.

For connected base spaces, the situation is quite different. In fact, we show that the hyperspace $K([0,1])$ of the closed unit interval admits no bi-Lipschitz embedding in any uniformly convex Banach space, more generally, it admits no bi-Lipschitz embedding in any round ball metric space, a concept introduced by Laakso [25]. Note that $K([0,1])$ is topologically equivalent with the Hilbert cube, as was shown by Schori and West [36], [37]. To prove our result, we exhibit within $K([0,1])$ a bi-Lipschitz copy of a certain series-parallel graph $G$ studied by Laakso [25], Lang–Plaut [26] and Lee–Mendel–Naor [27]. Laakso [25] showed that a variant of $G$ admits no bi-Lipschitz embedding in any round ball metric space; his argument can be adapted to apply to $G$. We constuct a planar iterated function complex (generalized iterated function system) parameterized by an uncountable collection of geodesic segments in $G$. The desired embedding of $G$ in $K([0,1])$ is obtained by choosing appropriate vertical slices of the attractors for this complex.

In section 2, we review some standard topics in metric geometry, give further history and motivation, and state our main results. In section 3 we consider countable compact metric spaces and finite-dimensional Euclidean targets, while section 4 contains the proof of the non-embeddability of the hyperspace of the unit interval in any round ball metric space. A concluding section 5 contains some questions for future study.
2. Review of metric geometry and statements of the main results

2.1. Notation and basic definitions. For a metric space $X = (X, d)$, we write $\text{diam} A$ (or $\text{diam}_e A$ in case the metric deserves to be mentioned) for the diameter of a set $A \subset X$ and $\text{dist}(A, B)$ for the distance between two nonempty sets $A, B \subset X$. We abbreviate $\text{dist}(A, x) = \text{dist}(A, \{x\})$ for $A \subset X$ and $x \in X$. We denote by $B(x, r)$ the closed ball in $X$ with center $x$ and radius $r$. A metric space $(X, d)$ is doubling if there exists a finite constant $M$ so that every ball $B(x, r)$ in $X$ can be covered by at most $M$ balls of radius $r/2$.

A modulus of continuity for a map $f : (X, d) \to (Y, d')$ between compact metric spaces is an increasing function $\omega : [0, \text{diam} X] \to [0, \infty)$ satisfying

$$d'(f(x), f(y)) \leq \omega(d(x, y))$$

for all $x, y \in X$. The map $f$ is $L$-Lipschitz, $L < \infty$, if it has a modulus of continuity of the form $\omega(t) = L \cdot t$, and is $\beta$-Hölder continuous, $0 < \beta \leq 1$, if it has a modulus of continuity of the form $\omega(t) = C \cdot t^\beta$ for some $C < \infty$. If the value of the constants $L$ or $\beta$ are not important, the terms Lipschitz or Hölder continuous are used. An $L$-Lipschitz map with $L < 1$ is called a contraction.

If $f$ is a homeomorphism and $f$ and $f^{-1}$ are both ($L$-)Lipschitz, $f$ is said to be ($L$-)bi-Lipschitz. If $f^{-1}$ is Hölder, $f$ is said to be co-Hölder.

A gauge function is a nondecreasing, right-continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$. Associated with each gauge function is the generalized Hausdorff $\varphi$-measure $\mathcal{H}^\varphi$ on $X$, defined for $E \subset X$ as $\mathcal{H}^\varphi(E) := \lim_{s \to 0} \inf \{\sum_i \varphi(\text{diam} E_i) : (E_i)_{i \in \mathbb{N}}, X = \cup_i E_i\}$. The case $\varphi_s(t) = t^s$ gives the classical Hausdorff measures $\mathcal{H}^s := \mathcal{H}^{t^s}$.

We write $s \vee t$, resp. $s \wedge t$, for the maximum, resp. minimum, of two real numbers $s$ and $t$.

2.2. Hyperspaces and the Hausdorff metric. The (compacta) hyperspace of a topological space $X$ is the space $K(X)$ of all nonempty compact subsets of $X$. When $X = (X, d)$ is a metric space, $K(X)$ is equipped with the Hausdorff metric

$$D(A, B) = D_d(A, B) := \inf \{\epsilon : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\}$$

$$= \max_{a \in A} \text{dist}(a, B) \vee \max_{b \in B} \text{dist}(b, A),$$

for $A, B \subset K(X)$, where $N_\epsilon(A) = \{x \in X : d(x, a) < \epsilon \text{ for some } a \in A\}$ denotes the $\epsilon$-neighborhood of $A$.

The hyperspace $K(X)$ of a complete (resp. compact) space $X$ is complete (resp. compact). See, e.g., [22, Chapter I]. Furthermore, $X$ embeds isometrically in $K(X)$ and $\text{diam}_D K(X) = \text{diam}_D X$. The map $A \mapsto \text{diam} A$ is a 2-Lipschitz map from $K(X)$ to $\mathbb{R}$. If $X$ is separable, then $K(X)$ is also separable. Indeed, if $Y$ is a countable dense subset of $X$, then the collection of all finite subsets of $Y$ is a countable dense subset of $K(X)$.

2.3. Hyperspaces of uniformly disconnected spaces. A metric $d$ on a space $X$ is called an ultrametric if $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$. A metric space $(X, d)$ is called uniformly disconnected if there exists an ultrametric $d'$ on $X$ so that the identity map from $(X, d)$ to $(X, d')$ is bi-Lipschitz. Uniformly disconnected spaces were introduced by David and Semmes [9, Chapter 15] using a different but equivalent definition.
As our first result, we observe that the hyperspaces of uniformly disconnected spaces admit bi-Lipschitz embeddings in Hilbert spaces.

**Proposition 2.4.** Let \((X, d)\) be a uniformly disconnected metric space. Then \(K(X)\) admits a bi-Lipschitz embedding in a Hilbert space.

The solution to the isometric embedding problem for Hilbert space targets was a major advance in twentieth century metric geometry. Work of Schoenberg \[34\], \[35\] and Blumenthal \[4\] provided an intrinsic characterization of those metric spaces which admit an isometric embedding in Hilbert space. Timan \[38\], Lemin \[28\] and Aschbacher et al. \[1\] gave independent proofs that every ultrametric space admits an isometric embedding in a Hilbert space. Thus every uniformly disconnected metric space admits a bi-Lipschitz embedding in a Hilbert space. Proposition 2.4 is an easy consequence of this fact.

**Proof of Proposition 2.4.** By the result of Timan, Lemin and Aschmacher et al., it suffices to prove that \(K(X)\) is uniformly disconnected. It is immediate from the definition that hyperspaces of ultrametric spaces are ultrametric. Furthermore, if \(f : X \to Y\) is a bi-Lipschitz map of metric spaces, then the induced map \(F_f : K(X) \to K(Y)\) given by \(F_f(A) = \{f(a) : a \in A\}\) is also bi-Lipschitz. The result follows.

Since hyperspaces of separable spaces are separable, and separable ultrametric spaces embed isometrically in \(\ell^2\), we also have the following

**Corollary 2.5.** If \((X, d)\) is separable and uniformly disconnected, then \(K(X)\) admits a bi-Lipschitz embedding in \(\ell^2\).

For example, the hyperspace of the standard Cantor set \(C\) admits a bi-Lipschitz embedding in \(\ell^2\). Observe that \(K(C)\) is perfect and totally disconnected, hence homeomorphic with \(C\). We do not know to what extent this topological equivalence can be made quantitative.

**Question 2.6.** What is the optimal modulus of continuity for a homeomorphism of \(K(C)\) onto \(C\)?

The characterization of metric spaces which admit a bi-Lipschitz embedding in a finite-dimensional Euclidean space is a major open problem in geometric analysis. The following theorem provides some infinite metric spaces whose hyperspaces admit such an embedding.

**Theorem 2.7.** Let \(X\) be a compact metric space containing at most \(N\) non-isolated points. Enumerate the (countable) set of isolated points \(I(X) = \{x_1, x_2, \ldots\}\) so that the sequence \(d_n := \text{dist}(x_n, X \setminus I(X))\) is non-increasing. Then

(a) there exists a Lipschitz embedding of \(K(X)\) in \(\mathbb{R}^{N+1}\),

(b) if

\[
    r^+ := \limsup_{n \to \infty} \frac{d_{n+1}}{d_n} < 1
\]

then the embedding in part (a) may be chosen to be co-Hölder,

(c) if \(r^+ < 1/2\), then the embedding in part (a) may be chosen to be bi-Lipschitz.
For example, the hyperspace of \( X = \{ a^n : n = 1, 2, \ldots \} \cup \{ 0 \}, \) \( a < 1, \) admits an embedding in \( \mathbb{R}^2 \) which is Lipschitz and co-Hölder. When \( a < \frac{1}{2} \) the embedding is bi-Lipschitz.

Pełczyński [32] characterized the hyperspaces of compact totally disconnected topological spaces containing a dense set of isolated points: for every such space \( X, \) the hyperspace \( K(X) \) is homeomorphic with the union of the Cantor set and the midpoints of the omitted intervals. Metric analogues of this result of Pełczyński remain to be established.

Observe that a condition such as (2.8) is necessary for the conclusion in Theorem 2.7. Indeed, for any prescribed Hausdorff gauge function \( \varphi \) there exists a countable metric space \((X, d)\) with precisely one non-isolated point, so that the generalized Hausdorff measure \( \mathcal{H}^\varphi(K(X)) \) is positive and finite. See, for example, Theorem 3.5 in [30].

We will prove Theorem 2.7 in section 3.

2.9. The hyperspace of the unit interval. Characterizing the hyperspace of \([0, 1]\) was a long-standing problem in the topology of hyperspaces. In 1938, Wojdysławski conjectured that \( K([0, 1]) \) was homeomorphic to the Hilbert cube \( Q = [0, 1]^\infty. \) The conjecture stood for over thirty years, until Schori and West proved it in 1972 [37]. Curtis and Schori [7] proved that the hyperspace of any nondegenerate Peano continuum is homeomorphic with \( Q. \)

The metric structure of \( K([0, 1]) \) was studied by Boardman and Goodey, who determined some gauge functions \( \varphi \) with respect to which \( K([0, 1]) \) has zero or infinite generalized Hausdorff measure [13], [14], [5]. The results of Boardman and Goodey were extended to the setting of general self-similar sets satisfying the open set condition by McClure [30].

A metric space \( X \) is called \textit{bi-Lipschitz homogeneous} if the group of bi-Lipschitz self-homeomorphisms of \( X \) acts transitively. In response to a question posed by Väisälä [40], Hohti [20] showed that the metric Hilbert cube \((Q, d_s) \subset c_0\) is bi-Lipschitz homogeneous if and only if

\[
\text{(2.10) } \sup_k \frac{s_k}{s_{k+1}} < \infty.
\]

Here \( s = (s_k)_{k \in \mathbb{N}} \) is a decreasing sequence of positive real numbers converging to zero, and \( d_s(x, y) = \max_k s_k |x_k - y_k|, \) \( x = (x_k), y = (y_k) \in Q. \) On the other hand, Hohti also showed in [20] that \( K([0, 1]) \) is not bi-Lipschitz homogeneous. It follows that \( K([0, 1]) \) is not bi-Lipschitz equivalent with \((Q, d_s)\) for any \( s \) satisfying (2.10).

Using a comparison principle for the cardinality of nets in bi-Lipschitz equivalent spaces, Hohti also demonstrated the bi-Lipschitz inequivalence of \( K([0, 1]) \) with \((Q, d)\) for some other metrics \( d. \) For example, \( K([0, 1]) \) and \((Q, \rho_s) \subset \ell^1\) are bi-Lipschitz inequivalent, where \( s = (s_k), s_k = 2^{-k}, \) and \( \rho_s(x, y) = \sum_k s_k |x_k - y_k|. \) The proof easily extends to the case \((Q, \rho_{s,p}) \subset \ell^p, 1 < p < \infty, \) where \( \rho_{s,p}(x, y) = (\sum s_k^p |x_k - y_k|^p)^{1/p}. \)

The following theorem further illustrates the intrinsic metric complexity of \( K([0, 1]) \) and significantly extends the preceding results of Hohti.

\textbf{Theorem 2.11.} There is no bi-Lipschitz embedding of \( K([0, 1]) \) into any round ball metric space.
Following Laakso [25], we say that a metric space \((X, d)\) is a round ball space if for every \(\varepsilon > 0\) there exists \(\delta = \delta(\varepsilon) > 0\) so that
\[
\text{diam } B(x, \left(\frac{1}{2} + \delta\right)d(x, y)) \cap B(y, \left(\frac{1}{2} + \delta\right)d(x, y)) \leq \varepsilon d(x, y)
\]
for all \(x, y \in X\). We call \(\delta(\varepsilon)\) the roundness function of \(X\). If \(X\) is a Banach space, then \(X\) is a round ball space if and only if \(X\) is uniformly convex [25]. Bi-Lipschitz embeddings in round ball metric spaces were studied by Laakso [25] in connection with strong \(A_\infty\) deformations of geometry and by Tyson and Wu [39] in connection with snowflake metric spaces and Assouad’s embedding theorem [2].

With no a priori control on the geometry of the target, it is not clear whether one can prove Theorem 2.11 by the techniques of [20]. Note also that generalized Hausdorff measures cannot be used to rule out the existence of such an embedding. Indeed, a classical result of Goodey [12] asserts that every infinite-dimensional Banach space has non-\(\sigma\)-finite Hausdorff measure for every gauge function \(\varphi\).

To prove Theorem 2.11, we use recent results on the bi-Lipschitz non-embeddability of certain series-parallel graphs due to Laakso, Lang–Plaut and Lee–Mendel–Naor. Consider the infinite graph \(G\), obtained as the Gromov–Hausdorff limit of the sequence of finite series-parallel graphs \(G_n\) shown in Figure 1. In rough terms, \(G_n\) is obtained by replacing each edge in \(G_{n-1}\) with a scaled copy of the graph \(G_1\). For the precise definition, see Definition 4.8. We equip \(G\) with the path metric.

Laakso [25] used a variant of this graph to answer a question of Heinonen and Semmes [19, Question 1] on the existence of a planar strong \(A_\infty\) weight (in the sense of David and Semmes [8], [9]) whose deformed geometry fails to admit a bi-Lipschitz embedding in any uniformly convex Banach space. Motivated by algorithmic problems in network theory and the failure of the Johnson–Lindenstrauss dimension reduction lemma in \(L^1\), Lee, Mendel and Naor [27] showed that \(G\) admits no bi-Lipschitz embedding in \(L^p\) for any \(1 < p \leq 2\). See also Lang and Plaut [26]. Following the terminology of [27], we call \(G\) the Laakso graph.

**Proposition 2.13.** The Laakso graph \(G\) does not admit a bi-Lipschitz embedding in any round ball metric space.

For the sake of completeness, we include a proof of Proposition 2.13 in the next section, following the argument given by Laakso [25, §2.5].

Theorem 2.11 is a consequence of Proposition 2.13 and the following theorem, which we prove in section 4.
Theorem 2.14. There is a bi-Lipschitz embedding of the Laakso graph $G$ in $K([0,1])$.

We do not know whether $K([0,1])$ admits a bi-Lipschitz embedding in $L^1$, or whether every bounded series-parallel graph admits a bi-Lipschitz embedding in $K([0,1])$. Gupta et al. [18] have shown that $\ell^1$ contains a 14-bi-Lipschitz copy of every series-parallel graph.

For additional questions and discussion, see section 5.

3. Proof of Theorem 2.7

Throughout this section, we let $X$ be a compact metric space and we denote by $I(X)$, resp. $L(X)$, the set of isolated, resp. limit, points of $X$. As in the statement of Theorem 2.7, we assume that $I(X) = \{x_1, x_2, \ldots\}$, where $d_n := \text{dist}(x_n, L(X))$ is non-increasing. For $x \in I(X)$, we define the *gap* of $X$ at $x$ to be

$$\text{gap}(X, x) := \inf\{d(x, y) : y \in X, x \neq y\}.$$  

Fix $\lambda > 2$ and define

$$\gamma_n := \min\{\lambda^{-n} \text{gap}(X, x_i) : i = 1, \ldots, n\}.$$  

Lemma 3.1. For each $n \in \mathbb{N}$, $\sum_{m>n} \gamma_m \leq \frac{1}{\lambda - 1} \gamma_n$.

Proof. For each $i = 1, \ldots, n$,

$$\sum_{m>n} \gamma_m \leq \sum_{m=n+1}^{\infty} \lambda^{i-m} \text{gap}(X, x_i) = \frac{1}{\lambda - 1} \lambda^{-n} \text{gap}(X, x_i)$$

and the result follows. \qed

The following theorem gives a more precise statement of part (a) of Theorem 2.7.

Theorem 3.2. Let $X$ be a countable compact metric space with $N$ limit points. Then there is a Lipschitz embedding $F$ of $K(X)$ in $\mathbb{R}^{N+1}$ whose inverse has a modulus of continuity

$$\omega(t) = t + 2\omega_0(\frac{\lambda - 1}{\lambda - 2} t),$$

where $\omega_0$ is any increasing function from $[0, \infty)$ to $[0, \infty)$ with $\omega_0(0) = 0$ and

$$d_n \leq \omega_0(\gamma_n)$$

for all $n$.

Proof. We denote points in $\mathbb{R}^{N+1}$ by $x = (x_0, \ldots, x_N)$, and use the maximum metric

$$\|x - y\|_{\infty} = \max_{i=0, \ldots, M} |x_i - y_i|, \quad x, y \in \mathbb{R}^{N+1}.$$  

Let $L(X) = \{z_1, \ldots, z_N\}$ and define $F : K(X) \to \mathbb{R}^{N+1}$,

$$F(A) = (f_0(A), \ldots, f_N(A)),$$

by

$$f_i(A) = \text{dist}(A, z_i), \quad i = 1, \ldots, N,$$
and
\[ f_0(A) = \sum_{n : x_n \in A} \gamma_n. \]

Observe that \( f_0(A) = f_0(A \cap I(X)) \) for all \( A \in K(X). \)

**Lemma 3.5.** Suppose that \( A \cap I(X) \neq B \cap I(X) \) for some \( A, B \in K(X). \) Then
\[
\frac{\lambda - 2}{\lambda - 1} \gamma_{n_1} \leq |f_0(A) - f_0(B)| \leq \frac{\lambda}{\lambda - 1} \gamma_{n_1} < 2 \gamma_{n_1},
\]
where \( n_1 = \min\{n : x_n \in A \triangle B\}. \)

Here \( A \triangle B := (A \setminus B) \cup (B \setminus A) \) denotes the symmetric difference of \( A \) and \( B. \)

**Proof of Lemma 3.5.** Without loss of generality assume that \( x_{n_1} \in A. \) By Lemma 3.1,
\[
|f_0(A) - f_0(B)| = \left| \gamma_{n_1} + \sum_{n > n_1 : x_n \in A \setminus B} \gamma_n - \sum_{n > n_1 : x_n \in B \setminus A} \gamma_n \right|
\geq \gamma_{n_1} - \sum_{n > n_1} \gamma_n \geq \frac{\lambda - 2}{\lambda - 1} \gamma_{n_1}.
\]

In a similar manner, we find that \( |f_0(A) - f_0(B)| \leq \gamma_{n_1} + \sum_{n > n_1} \gamma_n \leq \frac{\lambda}{\lambda - 1} \gamma_{n_1}. \)

Returning to the proof of Theorem 3.2, let \( A, B \in K(X), A \neq B. \) If \( A \cap I(X) = B \cap I(X) \) then \( f_0(A) = f_0(B). \) Otherwise, using Lemma 3.5 we find
\[
|f_0(A) - f_0(B)| \leq 2 \gamma_{n_1} \leq 2 \text{gap}(X, x_{n_1}) \leq 2D(A, B).
\]
On the other hand, for any \( i = 1, \ldots, N \) and any \( b \in B, \)
\[ f_i(A) = \text{dist}(A, z_i) \leq \text{dist}(A, b) + d(b, z_i) \leq D(A, B) + d(b, z_i) \]
so \( f_i(A) \leq D(A, B) + f_i(B). \) Reversing the roles of \( A \) and \( B \) shows that
\[
|f_i(A) - f_i(B)| \leq D(A, B),
\]
and combining (3.7) and (3.8) yields
\[
||F(A) - F(B)||_\infty \leq 2D(A, B).
\]

To compute a modulus of continuity for \( F^{-1}, \) let \( A, B \in K(X), A \neq B. \) If \( A \cap I(X) = B \cap I(X), \) then \( A \cap L(X) \neq B \cap L(X) \) and
\[
D(A, B) = \max_{a \in A} \text{dist}(a, B) \vee \max_{b \in B} \text{dist}(b, A) = \max_{a \in A'} \text{dist}(a, B) \vee \max_{b \in B'} \text{dist}(b, A),
\]
where \( A' = (A \setminus B) \cap L(X) \) and \( B' = (B \setminus A) \cap L(X). \) (Note that \( A' \cup B' \neq \emptyset. \) Thus
\[
D(A, B) \leq \max_{z_i \in A' \cup B'} |f_i(A) - f_i(B)| \leq ||F(A) - F(B)||_\infty \leq \omega(||F(A) - F(B)||_\infty),
\]
where \( \omega(t) \) is as in (3.3).
Suppose instead that \( A \cap \mathcal{I}(X) \neq B \cap \mathcal{I}(X) \) and let \( A'' = (A \setminus B) \cap \mathcal{I}(X) \) and \( B'' = (B \setminus A) \cap \mathcal{I}(X) \). Choose \( n_1 \) as in Lemma 3.5. For each \( a \in A'' \), \( \text{dist}(a, \mathcal{P}(X)) = d(a, z_i) \) for some \( z_i \in L(X) \). Hence \( f_i(A) \leq d(a, z_i) \leq d_{n_1} \) and
\[
\text{dist}(a, B) \leq d(a, z_i) + \text{dist}(z_i, B) \leq d_{n_1} + f_i(B),
\]
\[
\leq 2d_{n_1} + |f_i(B) - f_i(A)| \leq 2d_{n_1} + \|F(B) - F(A)\|_{\infty}.
\]
Similarly \( \text{dist}(b, A) \leq 2d_{n_1} + \|F(B) - F(A)\|_{\infty} \) for all \( b \in B'' \). It follows that
\[
D(A, B) = \max_{a \in A''} \text{dist}(a, B) + \max_{b \in B''} \text{dist}(b, A)
\leq 2d_{n_1} + \|F(B) - F(A)\|_{\infty}
\leq 2\omega_0(\gamma_{n_1}) + \|F(B) - F(A)\|_{\infty}
\leq 2\omega_0 \left( \frac{\lambda - 1}{\lambda - 2} |f_0(B) - f_0(A)| \right) + \|F(B) - F(A)\|_{\infty}
\leq \omega(\|F(B) - F(A)\|_{\infty}).
\]
We used Lemma 3.5 in the penultimate step.

Remark 3.12. The construction in the preceding proof was inspired by a paper of Reiter and Reiter [33], who gave an explicit form for the embedding in the theorem of Pełczyński in the setting of compact metric spaces with a single non-isolated point.

Proof of Theorem 2.7(b),(c). Assume that \( r^+ < 1 \) and choose \( r^+ < \rho < 1 \). If \( r^+ < \frac{1}{2} \) choose \( r^+ < \rho < \frac{1}{2} \). Then there exists \( n_2 \) so that \( d_{n+1} \leq \rho d_n \) for all \( n \geq n_2 \). Let \( Z = \{x_1, \ldots, x_{n_2}\} \) and let
\[
\varepsilon_Z = \min_{x_n \in Z} \text{gap}(X, x_n).
\]
The Lipschitz/Hölder constant for \( F^{-1} \) will depend on the parameters \( n_2 \) and \( \varepsilon_Z \).

Lemma 3.13. For all \( n > n_2 \), \( \text{gap}(X, x_n) \geq (1 - \rho)d_n \).

Proof. For \( n \in \mathbb{N} \),
\[
\text{gap}(X, x_n) = \inf \{d(x_i, x_n) : i \neq n\}
\geq (d_1 - d_n) \wedge \cdots \wedge (d_{n-1} - d_n) \wedge \inf \{d_n - d_i : i > n\}
\geq (d_{n-1} - d_n) \wedge (d_n - d_{n+1}).
\]
If \( n > n_2 \) then
\[
\text{gap}(X, x_n) \geq (1 - \rho)(d_{n-1} \wedge d_n) = (1 - \rho)d_n
\]
as desired.

Returning to the proof of Theorem 2.7, we consider two cases according to the value of \( \rho \).

Case 1. If \( \rho < \frac{1}{2} \), choose \( \lambda = 1/\rho > 2 \). Then \( \gamma_n = \min\{\rho^{n-i} \text{gap}(X, x_i) : i = 1, \ldots, n\} \). For \( n \leq n_2 \) we have the trivial estimate
\[
\gamma_n \geq \left( \frac{\rho^{n_2-1} \varepsilon_Z}{\text{diam} X} \right) d_n.
\]
For \( 1 \leq i \leq n_2 < n \) we estimate
\[
\rho^{n-i} \text{gap}(X, x_i) \geq \left( \frac{\rho^{n_2-i} \varepsilon_Z}{d_{n_2}} \right) \rho^{n-n_2} d_n \geq \left( \frac{\rho^{n_2-1} \varepsilon_Z}{\text{diam} X} \right) d_n.
\]
while for $n_2 < i \leq n$ we use Lemma 3.13 to estimate
\[
\rho^{n-i} \text{gap}(X, x_i) \geq (1 - \rho)\rho^{n-i}d_i \geq (1 - \rho)d_n.
\]
In all cases we obtain
\[
\gamma_n \geq \min\left\{ 1 - \rho, \frac{\rho^{n_2-1}\epsilon_Z}{\text{diam } X} \right\} d_n
\]
and deduce (3.4) with
\[
\omega_0(t) = \max\left\{ \frac{1}{1 - \rho}, \frac{\text{diam } X}{\rho^{n_2-1}\epsilon_Z} \right\} t.
\]

Case 2. If $\frac{1}{2} \leq \rho < 1$, choose $\lambda = 3$ and
\[
(3.14) \quad \alpha = \log(1/\rho)/\log 3.
\]
Then $\gamma_n^\alpha = \min\{\rho^{n-i}\text{gap}(X, x_i)^\alpha : i = 1, \ldots, n\}$. For $n \leq n_2$ we have the trivial estimate
\[
\gamma_n^\alpha \geq \left( \frac{\rho^{n_2-1}\epsilon_Z^\alpha}{\text{diam } X} \right) d_n.
\]

As in the preceding case we deduce that
\[
\rho^{n-i} \text{gap}(X, x_i)^\alpha \geq \begin{cases} 
(\text{diam } X)^{-1}\rho^{n_2-1}\epsilon_Z^\alpha d_n, & 1 \leq i \leq n_2 < n, \\
(\text{diam } X)^{\alpha-1}(1 - \rho)^\alpha d_n, & n_2 < i \leq n.
\end{cases}
\]
Hence
\[
\gamma_n^\alpha \geq \min\left\{ \frac{(1 - \rho)^\alpha}{(\text{diam } X)^{1-\alpha}}, \frac{\rho^{n_2-1}\epsilon_Z^\alpha}{\text{diam } X} \right\} d_n
\]
so (3.4) holds with
\[
\omega_0(t) = \max\left\{ \frac{(\text{diam } X)^{1-\alpha}}{(1 - \rho)^\alpha}, \frac{\text{diam } X}{\rho^{n_2-1}\epsilon_Z^\alpha} \right\} t^\alpha.
\]
Using Theorem 3.2, we conclude that $F^{-1}$ is $\alpha$-Hölder continuous if $\frac{1}{2} \leq \rho < 1$, where $\alpha$ is the value in (3.14), and that $F^{-1}$ is Lipschitz continuous if $\rho < \frac{1}{2}$. \hfill \Box

Remark 3.15. We do not know whether there exists a space $X$, with $L(X)$ infinite, so that $K(X)$ admits a bi-Lipschitz, or Lipschitz and co-Hölder, embedding in a finite-dimensional Euclidean space.

4. Proof of Theorem 2.14

4.1. Iterated function systems and iterated function complexes. An iterated function system (or IFS) is a finite collection of contractions of a complete metric space. The basic theory of IFS’s can be found in standard textbooks of fractal geometry, e.g., [11], [29], [10] or [23]. We briefly recall the basic theory.

Associated to each IFS $\mathcal{F}$ on a complete metric space $X$ is a unique nonempty compact set $A = A(\mathcal{F}) \in K(X)$, called the attractor of $\mathcal{F}$, characterized by the relation $A = F(A)$, where
\[
(4.2) \quad F(S) := \bigcup_{f \in \mathcal{F}} f(S).
\]
Indeed, the map $F$ defined in (4.2) is a contraction of $K(X)$, and the attractor $A$ is the unique fixed point for $F$. It is easy to see that $A = \bigcap_{m=1}^{\infty} F^{(m)}(T)$ for any $T \in K(X)$ with $F(T) \subset T$, where $F^{(m)}$ denotes the $m$-fold composition of $F$.

Let $M$ be the cardinality of $\mathcal{F}$. For $m \in \mathbb{N}$, we let $W_M^m = \{1, 2, \ldots, M\}^m$ be the space of words of length $m$ on $M$ letters. We abbreviate $W_M := W_M^1$. For $m = 0$ we set $W_M^0 = \{\emptyset\}$. We let $W_M^* = \bigcup_{m \geq 1} W_M^m$ be the space of all finite words, and $\Sigma_M = \{1, 2, \ldots, M\}^\mathbb{N}$ be the space of infinite words on $M$ letters (commonly referred to as the shift space or symbol space). We denote elements of $W_M^m$, $m \in \mathbb{N}$, and $\Sigma_M$ by concatenation of letters, i.e., $w = w_1w_2 \cdots w_m \in W_M^m$ or $w = w_1w_2 \cdots \in \Sigma_M$. For $v \in W_M^*$ and $w \in W_M^* \cup \Sigma_M$ we denote by $vw$ the word obtained by adjoining $v = v_1 \cdots v_m$ to the left of $w = w_1w_2 \cdots$, i.e., $vw = v_1 \cdots v_m w_1w_2 \cdots$. Conversely, for $w \in \Sigma_M$ and $m \in \mathbb{N}$ we denote by $[w]_m$ the word in $W_M^m$ consisting of the first $m$ letters of $w$; thus $w = [w]_m w'$ for some $w' \in \Sigma_M$.

For each IFS $\mathcal{F} = \{f_1, \ldots, f_M\}$ there is a canonical surjection $\pi_\mathcal{F} : \Sigma_M \to A(\mathcal{F})$ given by

$$\{\pi_\mathcal{F}(w_1w_2 \cdots)\} = \bigcap_{m=1}^{\infty} f_{w_1} \circ \cdots \circ f_{w_m}(T),$$

where $T$ is any element of $K(X)$ with $F(T) \subset T$. We write $f_w = f_{w_1} \circ \cdots \circ f_{w_m}$ for any $w = w_1 \cdots w_m \in W_M^*.$

We now introduce a generalization of the notion of IFS.

**Definition 4.3.** Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be a finite collection of iterated function systems on a complete metric space $X$. For $v = v_1v_2 \cdots \in \Sigma_r$, let $\mathcal{F}_v = (\mathcal{F}_{v_1}, \mathcal{F}_{v_2}, \ldots)$. The collection of sequences $\{\mathcal{F}_v\}_{v \in \Sigma_r}$ is called an iterated function complex (or IFC).

Associated to each rank $r$ IFC $\{\mathcal{F}_v\}_{v \in \Sigma_r}$ is a family of nonempty compact sets $\{A^v\}_{v \in \Sigma_r}$ of $X$, the attractor of $\{\mathcal{F}_v\}$, characterized by the identity

$$A^v = \bigcap_{m=1}^{\infty} F_{v_1} \circ \cdots \circ F_{v_m}(T)$$

for any $T$ such that $F_i(T) \subset T$ for all $i$. Here $F_i$, $i = 1, \ldots, r$, denotes the set function for $\mathcal{F}_i$ defined in (4.2). Indeed, the set functions $F_1, \ldots, F_r$ form an IFS on $K(X)$ whose attractor $A \in K(K(X))$ is precisely the collection $\{A^v\}$.

Observe that the set $A^\mathbf{7}$ defined by (4.4) with $\mathbf{7} = (i, i, \ldots)$ coincides with the attractor $A^i$ for the IFS $\mathcal{F}_i$.

4.5. **The Laakso graph.** In this subsection, we describe the Laakso graph $G$ in further detail, first as a quotient of the product space $[0, 1] \times \Sigma_2$, and then as the attractor of a self-similar IFS.

We denote by $I_0 = [0, 1]$ the unit interval in $\mathbb{R}$, which we view as the attractor for the IFS $\mathcal{H} = \{h_1, h_2, h_3, h_4\}$, where

$$h_i : \mathbb{R} \to \mathbb{R}, \quad h_i(x) = (x + i - 1)/4.$$  

The symbolic coding map $\pi_{\mathcal{H}} : \Sigma_4 \to I_0$ corresponds to the representation of elements of $I_0 = [0, 1]$ in base four. We let $C$ be the $\frac{1}{4}$ Cantor set, i.e., the invariant set for the IFS
\{h_1, h_4\}. For \(m \in \mathbb{N}\), let

\[
O_m := \bigcup_{j=0}^{4^{m-1}-1} ((4j+1)/4^m, (4j+3)/4^m).
\]

The union of the open sets \(O_m\) is the complement of \(C\) in \(I_0\).

**Definition 4.8.** The **Laakso graph** is the space \(G = I_0 \times \Sigma_2 / \sim\), where \((t, v) \sim (t, v')\) if and only if \(t \in I_0 \setminus \bigcup \{O_m : v_m \neq v'_m\}\).

**Example 4.9.** Let \(T = 111 \cdots\) and \(Z = 222 \cdots\). Then \((t, T) \sim (t, Z)\) if and only if \(t \in C\).

We denote the equivalence class of \((t, v)\) in \(G\) by \([t, v]\), and equip \(G\) with the canonical quotient path metric, defined as follows:

\[
d([t, v], [t', v']) = \inf \sum_{j=1}^{k} |t(j) - t(j-1)|,
\]

where the infimum is taken over all finite sequences

\[
(t, v) = (t(0), v(0)), (t(1), v(1)), \ldots, (t(k), v(k)) = (t', v')
\]

satisfying \([t(j), v(j-1)] = [t(j), v(j)]\) for all \(j = 1, \ldots, k\). (Compare the “wormhole metrics” of [24] or [25].) Then \(G\) is a compact geodesic doubling metric space. The maps \(\gamma_v : I_0 \to G\), \(v \in \Sigma_2\), given by \(\gamma_v(t) = [t, v]\) are isometric embeddings, defining an uncountable family of geodesic segments in \(G\) joining the two endpoints.

The space \(G\) may be viewed as the attractor for a self-similar IFS of cardinality six. To simplify the description, we use an alternate notation for the alphabet. Thus let

\[
U = \{1, (2, 1), (2, 2), (3, 1), (3, 2), 4\}.
\]

The spaces \(U^m, m \in \mathbb{N}\), \(U^*\) and \(\Sigma_U\) of words of length \(m\), finite words, and infinite words are defined in the usual manner.

To each \(u \in U\) there corresponds a set \(S_u \subset G\) which is the image of \(G\) under a contractive similarity \(g_u\) with scale factor 1/4. These first-order similarity pieces are indicated in Figure 1. Thus \(G\) is the attractor for the IFS \(\mathcal{G} = \{g_u : u \in U\}\) with symbol space \(\Sigma_U\). As usual, we write \(g_u = g_{u_1} \circ \cdots \circ g_{u_m}\) and \(S_u = g_u(G)\) for \(u = u_1 \cdots u_m \in U^m\).

The following diagram commutes:

\[
\begin{array}{ccc}
\Sigma_4 \times \Sigma_2 & \xrightarrow{\pi_{4} \times \text{id}} & I_0 \times \Sigma_2 \\
\downarrow & & \downarrow \\
\Sigma_U & \xrightarrow{\pi_{g}} & G \\
\downarrow & & \downarrow \\
\Sigma_4 & \xrightarrow{\pi_{\gamma}} & I_0.
\end{array}
\]
Here the maps from $\Sigma_4 \times \Sigma_2$ to $\Sigma_U$ and from $\Sigma_U$ to $\Sigma_4$ are given, in each coordinate, by the following maps from $W_4 \times W_2$ to $U$ and from $U$ to $W_4$:

\[
\begin{align*}
(1, 1), (1, 2) & \rightarrow 1 \rightarrow 1 \\
(2, 1) & \rightarrow (2, 1) \searrow 2 \\
(2, 2) & \rightarrow (2, 2) \nearrow 3 \\
(3, 1) & \rightarrow (3, 1) \\
(3, 2) & \rightarrow (3, 2) \nearrow 4 \\
(4, 1), (4, 2) & \rightarrow 4 \rightarrow 4.
\end{align*}
\]

The map from $I_0 \times \Sigma_2$ to $G$ in (4.10) is the quotient map $(t, v) \mapsto [t, v]$, while the map from $G$ to $I_0$ is projection into the first coordinate $[t, v] \mapsto t$. Finally, $\pi_H : \Sigma_4 \rightarrow I_0$ and $\pi_G : \Sigma_U \rightarrow G$ denote the symbolic coding map for the IFS's $H$ and $G$.

**Sketch of the proof of Theorem 2.14:** In subsection 4.28, we will construct a specific rank two iterated function complex $\{F^v\}_{v \in \Sigma_2}$ in the plane. The essential feature of the attractor $\{A^v\}_{v \in \Sigma_2}$ for this IFC is the equivalence

\[
(t, v) \sim (t', v') \iff A^v_x = A^{v'}_x,
\]

which guarantees that the map $\Phi : G \rightarrow K(I_0)$ given by

\[
\Phi([t, v]) = A^v_x
\]

is well defined. We will prove that

\[
\Phi \circ \gamma_v : I_0 \rightarrow K(I_0)
\]

is an isometry for each $v \in \Sigma_2$.

We now prove Proposition 2.13. Compare the proof in [25, §2.5].

**Proof of Proposition 2.13.** Let $(Y, d')$ be a round ball space with roundness function $\delta(\epsilon)$, and suppose that $F : G \rightarrow Y$ is $L$-bi-Lipschitz. For each $x, y \in G$, let $L(x, y) := d'(F(x), F(y))/d(x, y)$ be the Lipschitz constant of $f|_{x,y}$. Let $a = S_1 \cap S_{(2,1)} \cap S_{(2,2)}$, $b = S_{(3,1)} \cap S_{(3,2)} \cap S_4$, $c = S_{(2,1)} \cap S_{(3,1)}$ and $d = S_{(2,2)} \cap S_{(3,2)}$. Observe that $\{c, d\} \subset B(a, 1/4) \cup B(b, 1/4)$ and $d(c, d) = 1/2$. Then $d'(F(c), F(d)) \geq L^{-2}d'(F(a), F(b))$. From the definition of round ball spaces, we conclude that

\[
\max\{L(a, c), L(a, d), L(c, b), L(d, b)\} \geq (1 + \delta(L^{-2}))L(a, b).
\]

Each pair of points on the left hand side coincides with the endpoints of one of the first-order similarity pieces $S_u$, $u \in U$. Thus the endpoints $x$ and $y$ of one of these pieces satisfy $L(x, y) \geq (1 + \delta(L^{-2}))L(a, b)$.

The same argument can be applied to points $a_u = S_{u,1} \cap S_{u, (2, 1)} \cap S_{u, (3, 1)}$ and $b_u = S_{u, (2, 2)} \cap S_{u, (3, 2)} \cap S_{u, 4}$ for any $u \in U^m$ and $m \in \mathbb{N}$. For each such $u$ and $m$ there exists a similarity piece of order $m + 1$ with endpoints $x_u$ and $y_u$ so that $L(x_u, y_u) \geq (1 + \delta(L^{-2}))L(a_u, b_u)$. 
Note that \( \sum_{m=1}^{\infty} \sum_{u \in \{1,4\}^m} (b_u - a_u) = 1 \). Hence \( L(a_u, b_u) \geq L(0,1) \) for some \( u \in \{1,4\}^m \), and so
\[
L(x_{u'}, y_{u'}) \geq (1 + \delta(L^{-2}))L(0,1)
\]
for the endpoints \( x_{u'}, y_{u'} \) of some similarity piece \( S_{u'} \) of \( G \). By self-similarity, \( S_{u'} \) contains a further similarity piece \( S_{u''} \) with endpoints \( x_{u''}, y_{u''} \) so that
\[
L(x_{u''}, y_{u''}) \geq (1 + \delta(L^{-2}))^2L(0,1).
\]
Continuing in this fashion, we eventually contradict the \( L \)-bi-Lipschitz character of \( F \). \( \square \)

4.16. \textbf{A lemma on geodesic segments in} \( K(I_0) \). The following lemma provides a sufficient condition for the vertical slices of a compact set \( A \subset I_0 \times I_0 \) to sweep out a geodesic in \( K(I_0) \).

We denote by \( Q_0 = I_0 \times I_0 \) the unit square in \( \mathbb{R}^2 \). For a compact set \( A \subset \mathbb{R}^2 \), we write \( A_t = \{ x : (t, x) \in A \} \) for the vertical slice of \( A \) with abscissa \( t \). Finally, for \( p = (t, x) \in Q_0 \), we write
\[
X(p) = \{ p' = (t', x') \in Q_0 : |t' - t| \geq |x' - x| \}.
\]

\( X(p) \) is the union of two antipodal sectors with opening angle \( \pi/4 \) centered at \( p \).

\textbf{Lemma 4.17.} For a compact set \( A \subset Q_0 \), the map \( t \mapsto A_t \) is an isometric embedding of \( I_0 \) in \( K(I_0) \) provided
\[
(4.18) \quad t \in A_t \subset [0,t] \quad \forall t \in I_0
\]
and
\[
(4.19) \quad A \subset \bigcup_{x \in A_t} X(t, x) \quad \forall t \in I_0.
\]

Condition (4.18) says that \( A \) lies in the triangle
\[
(4.20) \quad T_0 = \{ (t, x) : 0 \leq x \leq t \leq 1 \}
\]
and contains the diagonal \( \{ (t, t) : 0 \leq t \leq 1 \} \), while condition (4.19) says that whenever an open vertical segment \( J \) is in the complement of \( A \), then the open square with diagonal \( J \) is also in the complement of \( A \).

\textbf{Proof.} Let \( 0 \leq t' < t \leq 1 \). Condition (4.18) implies that
\[
(4.21) \quad D(A_t, A_{t'}) \geq \text{dist}(t, t') \geq |t - t'|.
\]
On the other hand, condition (4.19) implies that \( \{ t \} \times A_t \subset \bigcup_{x' \in A_t} X(x', t') \) and the corresponding statement with the roles of \( (t, x) \) and \( (t', x') \) reversed. Consequently
\[
(4.22) \quad D(A_t, A_{t'}) \leq |t - t'|.
\]
Inequalities (4.21) and (4.22) imply that \( t \mapsto A_t \) is an isometric embedding. \( \square \)

\textbf{Remark 4.23.} Inequality (4.21) can be generalized as follows: if \( A \) and \( A' \) are compact sets in \( Q_0 \) satisfying (4.18) and (4.19), then \( D(A_t, A'_{t'}) \geq |t - t'| \) for all \( t, t' \in I_0 \). The proof is the same.
Two planar iterated function systems. We begin by defining two planar IFS's corresponding to the digits in the alphabet \( W_2 \).

The 4-adic subsquares of \( Q_0 \) are the sixteen squares \( [(i - 1)/4, i/4] \times [(j - 1)/4, j/4], \), \( i, j = 1, 2, 3, 4 \).

The IFS \( \mathcal{F}_1 \) consists of nine planar contractive maps. Eight of these maps are contractive similarities, with scale factor \( 1/4 \), while the ninth is a degenerate affine contraction, with scale factor \( 1/4 \) in one direction and zero in the other direction. These nine contractions are indicated pictorially in Figure 2(a); they are the eight contractive maps of \( \mathbb{R}^2 \) which send the ordered triple \( A, B, C \) of vertices of \( Q_0 \) to each of the ordered triples \( a, b, c \) of vertices of the specified 4-adic subsquares of \( Q_0 \), together with the degenerate map sending \( A \) and the pair \( B, C \) to the endpoints \( a \) and \( b \) of the line segment labelled “8”. We label these maps \( f^j_{1}, j = 1, \ldots, 9 \), as indicated in Figure 2(a).

The IFS \( \mathcal{F}_2 \) consists of ten contractive similarities, each with scale factor \( 1/4 \). These similarities \( f^j_{2}, j = 1, \ldots, 10 \), are indicated pictorially in Figure 2(b).

![Figure 2. Similarities comprising (a) \( \mathcal{F}_1 \), (b) \( \mathcal{F}_2 \).](image)

**Remark 4.25.** Observe that \( \mathcal{F}_2 \) contains two contractions \( f^5_{2} \) and \( f^8_{2} \) mapping into a common subsquare. However, both \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) satisfy the open set condition [11, §9.2].

**Remark 4.26.** Note that the maps in the first and last columns coincide: \( f^j_{1} = f^j_{2} \) for \( j = 1, 4, 7 \) and \( f^9_{1} = f^{10}_{2} \). In the middle two columns, the set of maps is not the same: \( \{f^2_{1}, f^5_{1}\} \neq \{f^2_{2}, f^5_{2}, f^8_{2}\} \) and \( \{f^3_{1}, f^6_{1}, f^8_{1}\} \neq \{f^3_{2}, f^6_{2}, f^9_{2}\} \).

**Remark 4.27.** Figure 3 shows the attractors \( A^i \) for the IFS’s \( \mathcal{F}_i \), \( i = 0, 1 \). It is clear that conditions (4.18) and (4.19) hold for these attractors. It follows from the previous remark and the self-similarity of the constructions that \( A^1(m)_t = A^2(m)_t \) if and only if \( t \in I_0 \setminus \bigcup_{m=1}^{m} O_m \) for each \( m \in \mathbb{N} \), where \( A^i(m) \) is the image of \( T_0 \) under \( m \) iterates of \( F^i \), \( i = 1, 2 \). Hence \( A^1_t = A^2_t \) if and only if \( t \in C \), as asserted in (4.12) (see Example 4.9).

**A planar iterated function complex.** We now consider the rank two IFC \( \{\mathcal{F}_v\}_{v \in \Sigma_2} \) defined by \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \). To aid the reader, we describe the structure of the attractor of this
IFC in greater detail. Given an infinite word \( v = v_1 v_2 \cdots \in \Sigma_2 \), let \( W_v^m = W_{v_1} \times \cdots \times W_{v_m} \) and \( f^w_v = f^w_{v_1} \circ \cdots \circ f^w_{v_m} \) for all \( w \in W_v^m \), and let

\[
F_v^m(S) := \bigcup_{w \in W_v^m} f^w_v(S).
\]

The attractor for \( \{F_v \}_{v \in \Sigma_2} \) is the family of compact sets \( \{A^v \}_{v \in \Sigma_2} \), where \( A^v = \bigcap_{m=1}^{\infty} F_v^m(T_0) \). When \( v = \mathbf{1} \), \( A^v \) coincides with the attractor \( A^1 \) in Figure 3(a); when \( v = \mathbf{2} \), \( A^v \) coincides with \( A^2 \) in Figure 3(b). For any \( v \in \Sigma_2 \), \( A^v \) is the Hausdorff limit of the sequence \( A^v(m) := F_{v_1} \circ \cdots \circ F_{v_m}(T_0) \).

We now extend Remark 4.27 to the full IFC.

**Proposition 4.30.** For each \( v \in \Sigma_2 \), \( A^v \) satisfies (4.18) and (4.19). Furthermore, (4.12) holds.

**Proof.** Since \( F_i(Q_0) \subset T_0 \) and \( \mathcal{F}_i \supset \{f^j_i : j = 1, 2, 3, 4 \} \) for each \( i = 1, 2 \), (4.18) holds. To prove (4.19), let \( U_1, U_2 \) be the open sets shaded in Figure 2(a),(b). Then

\[
A^v(m) = \text{int}(Q_0) \setminus \bigcup_{i=1}^m \bigcup_{w \in W_v^i} f^w_v(U_{v_i})
\]

for each \( m \), where \( \text{int}(Q_0) \) denotes the interior of \( Q_0 \) and \( \overline{S} \) denotes the closure of a set \( S \). It follows that (4.19) is satisfied for each \( A^v(m) \), \( m \in \mathbb{N} \), and hence (4.19) is satisfied for \( A^v = \bigcap_m A^v(m) \).

To see why (4.12) holds, observe that Remark 4.26 guarantees that

\[
A^v(m)_t = A^{v'}(m)_t \iff t \in I_0 \setminus \bigcup_{j=1, \ldots, m: v_j \neq v'_j} O_j
\]

for all \( v, v' \in \Sigma_2 \) and \( m \in \mathbb{N} \), where \( O_m \) is the open set defined in (4.7). Then (4.12) follows by intersecting these conditions over all \( m \) and using the definition of the equivalence relation on \( I_0 \times \Sigma_2 \) (see Definition 4.8).

**Corollary 4.31.** The map \( \Phi : G \to K(I_0) \) given by \( \Phi([t, v]) = A^v_t \) is well-defined.
4.32. **Subsets and supersets of the sets** \( A^v \): In this section we prove two technical lemmas describing certain distinguished subsets and supersets of the vertical slices \( A^v \). These lemmas will play a key role in the proof of Theorem 2.14 by yielding lower bounds for the Hausdorff distance between slices \( A^v \) and \( A^{v'} \).

We begin by observing the common elements of Figure 2(a) and (b). Figure 4 shows a set of line segments \( \Gamma \subset Q_0 \) which are common to these two figures. Explicitly,

\[
\Gamma = \bigcup_{j=1}^{7} f_j^1(\Delta) \cup f_3^0(\Delta) = \bigcup_{j=1}^{8} f_j^2(\Delta) \cup f_2^{10}(\Delta),
\]

where \( \Delta := \{ (t, t) : t \in I_0 \} \) denotes the principal diagonal of \( I_0 \). Thus \( \Gamma \subset F_i(\Delta) \) for \( i = 1, 2 \), whence \( \Gamma \subset A^v \) for any \( v \in \Sigma_2 \). In fact,

\[
F^m_v(\Gamma) \subset A^v
\]

for all \( m \in \mathbb{N} \), where \( F^m_v \) is the set function defined in (4.29).

![Figure 4. The set \( \Gamma \)](image)

Recall our notation \([u]_m\) for the element of \( U^m \) consisting of the first \( m \) letters of \( u \in \Sigma_U \). For each \( m \in \mathbb{N} \), the projection of \( S_{[u]_m} \subset G \) to \( I_0 \) (via the map in (4.10)) is an interval \([a_{u,m}, b_{u,m}]\) with

\[
b_{u,m} - a_{u,m} = \text{diam} S_{[u]_m} = 4^{-m}.
\]

If \([t, v] = \pi_G(u)\), then

\[
a_{u,m} + \frac{i(u_{m+1}) - 1}{4} \cdot 4^{-m} \leq t \leq a_{u,m} + \frac{i(u_{m+1})}{4} \cdot 4^{-m},
\]

where \( i : U \to W_4 \) is as in (4.11).

In our first lemma, we describe some subsets of the slices \( A^v_i \). Let \( h_i, i = 1, 2, 3, 4 \) be the maps from (4.6).

**Lemma 4.36.** Let \( u \in \Sigma_U \) with \( \pi_G(u) = [t, v] \) and let \( m \in \mathbb{N} \).

(i) If \( u_{m+1} = (3, 2) \), then

\[
A^v_i \supset h_{i(u_1)} \circ \cdots \circ h_{i(u_m)} \circ h_1(\Gamma_s),
\]

where \( s = (a_{u,m} + \frac{3}{4} \cdot 4^{-m} - t)/\frac{1}{4} \cdot 4^{-m} \).

(ii) If \( u_{m+1} = (2, 2) \), then

\[
A^v_i \supset h_{i(u_1)} \circ \cdots \circ h_{i(u_m)} \circ h_1(\tilde{\Gamma}_s),
\]

where \( s = (t - a_{u,m} - \frac{1}{4} \cdot 4^{-m})/\frac{1}{4} \cdot 4^{-m} \) and \( \tilde{\Gamma} = \Gamma \cup f^1_1(\Gamma) \cup f^1_0(\Gamma) \cup \cdots \).
To each \( u \in U \), we associate a collection \( C(u) \) of elements of \( \mathcal{F}^1 \cup \mathcal{F}^2 \) as follows:

\[
\begin{align*}
C(1) &= \{ f_1^1 \}, & C(2,1) &= \{ f_1^2, f_1^5 \}, & C(3,1) &= \{ f_1^3, f_1^6, f_1^8 \}, \\
C(2,2) &= \{ f_2^2, f_2^5, f_2^8 \}, & C(3,2) &= \{ f_2^3, f_2^6, f_2^9 \}, & C(4) &= \{ f_2^4, f_2^6, f_2^{10} \}.
\end{align*}
\]

Observe that each collection \( C(u) \) contains precisely one diagonal contraction, namely \( H_{i(u)} := h_{i(u)} \times h_{i(u)} \). (By a diagonal contraction we mean one of the maps \( f_i^n \), \( i = 1, 2, 3, 4 \).)

**Proof of Lemma 4.36.** Let \( u \in \Sigma_U \) be as in the statement of the lemma. From (4.33), we deduce that

\[
A^v \supset H_{i(u_1)} \circ \cdots \circ H_{i(u_m)}(F_{v_{m+1}}(\Gamma)).
\]

If \( u_{m+1} = (3,2) \), then

\[
A^v \supset [H_{i(u_1)} \circ \cdots \circ H_{i(u_m)} \circ f_2^g(\Gamma)]_t = h_{i(u_1)} \circ \cdots \circ h_{i(u_m)} \circ h_1(\Gamma_s)
\]

where \( s = (a_{u,m} + \frac{3}{4} \cdot 4^{-m} - t) / (\frac{1}{4} \cdot 4^{-m}) \). (Note that \( f_2^g(t,x) = (h_3(1-t), h_1(1-x)) \).)

Similarly, if \( u_{m+1} = (2,2) \), then

\[
A^v \supset [H_{i(u_1)} \circ \cdots \circ H_{i(u_m)} \circ f_2^g(\Gamma)]_t = h_{i(u_1)} \circ \cdots \circ h_{i(u_m)} \circ h_1(\Gamma_s)
\]

where \( s = (t - a_{u,m} - \frac{1}{4} \cdot 4^{-m}) / (\frac{1}{4} \cdot 4^{-m}) \). (Note that \( f_2^g(t,x) = (h_2(t), h_1(x)) \).)

Next, we give some supersets of the slices \( A^v \).

**Lemma 4.37.** Let \( u \in \Sigma_U \) with \( \pi_G(u) = [t,v] \) and let \( m \in \mathbb{N} \).

(i) If \( u_{m+1} = (3,1) \), then \( A^v \subset [0,a_{u,m}] \cup [a_{u,m} + \frac{3}{4} \cdot 4^{-m}, t] \).

(ii) If \( u_{m+1} = (2,1) \), then \( A^v \subset [0,a_{u,m}] \cup [t - \frac{1}{4} \cdot 4^{-m}, t] \).

**Proof.** Let \( m \in \mathbb{N} \). By construction,

\[
A^v = F^m_v(A^v) \subset F^m_v(T_0).
\]

If \( u_{m+1} \in \{(3,1), (2,1)\} \), then \( v_{m+1} = 1 \) and

\[
A^v \subset F^m_v(F_1(T_0)) \subset F^m_v(T_0 \setminus U_1),
\]

where \( U_1 \) denotes the open set from the proof of Proposition 4.30. Separating the diagonal and non-diagonal elements gives

\[
A^v \subset I_0 \times [0,a_{u,m}] \cup H_{i(u_1)} \circ \cdots \circ H_{i(u_m)}(T_0 \setminus U_1)
\]

whence

\[
A^v \subset [0,a_{u,m}] \cup [H_{i(u_1)} \circ \cdots \circ H_{i(u_m)}(T_0 \setminus U_1)]_{t} = [0,a_{u,m}] \cup h_{i(u_1)} \circ \cdots \circ h_{i(u_m)}((T_0 \setminus U_1)_s),
\]

where \( s = (t - a_{u,m}) / 4^{-m} \). If \( u_{m+1} = (2,1) \), then by (4.35) we have \( \frac{1}{2} \leq s \leq \frac{1}{2} \) and

\[
A^v \subset [0,a_{u,m}] \cup h_{i(u_1)} \circ \cdots \circ h_{i(u_m)}([s - \frac{1}{4}, s]) = [0,a_{u,m}] \cup [t - \frac{1}{4} \cdot 4^{-m}, t];
\]

if \( u_{m+1} = (3,1) \), then by (4.35) we have \( \frac{1}{2} \leq s \leq \frac{3}{4} \) and

\[
A^v \subset [0,a_{u,m}] \cup h_{i(u_1)} \circ \cdots \circ h_{i(u_m)}([\frac{1}{4}, s]) = [0,a_{u,m}] \cup [a_{u,m} + \frac{1}{4} \cdot 4^{-m}, t].
\]

\( \square \)
Proof of Theorem 2.14. By Corollary 4.31 the map $\Phi : G \to K(I_0)$ given by $\Phi([t, v]) = A_t^v$ is well-defined. The assertion in (4.15) follows from Proposition 4.30. To prove (4.14) we will show that
\begin{equation}
D(A_t^v, A_{t'}^{v'}) \leq K_1 d(p, p')
\end{equation}
with $K_1 = 4$, and
\begin{equation}
d(p, p') \leq K_2 D(A_t^v, A_{t'}^{v'})
\end{equation}
with $K_2 = 33$, for all points $p = [t, v]$ and $p' = [t', v']$ in $G$.

Remark 4.40. We do not know the best bi-Lipschitz constants $K_1$ and $K_2$ for the embedding $\Phi$. Let $p = [1/2, 1v]$ and $p' = [1/2, 2v']$ for arbitrary $v, v' \in \Sigma_2$. Then
\[
\left\{ \frac{1}{4}, \frac{1}{2} \right\} \subset \Phi(p) = A_{1/2}^v \subset \left[ \frac{1}{4}, \frac{1}{2} \right]
\]
and
\[
\left\{ 0, \frac{1}{4}, \frac{1}{2} \right\} \subset \Phi(p') = A_{1/2}^{v'} \subset \left[ 0, \frac{1}{2} \right],
\]
so $d(p, p') = \frac{1}{2}$ and $D(\Phi(p), \Phi(p')) = \frac{1}{4}$, which shows that $K_2$ cannot be smaller than 2.

Returning to the proof of the theorem, let $p = [t, v]$ and $p' = [t', v']$ be distinct elements of $G$. Choose a word $[u]_m \in U^m$ of maximal length $m$ so that $p, p' \in S_{[u]_m}$. By changing the representatives of $p$ and $p'$ if necessary, we may assume that $v_j = v'_j$ for $j = 1, \ldots, m$. We begin by noting that
\begin{equation}
|t - t'| \leq D(A_t^v, A_{t'}^{v'}) \leq \text{diam } S_{[u]_m} = 4^{-m};
\end{equation}
see Remark 4.23 for the lower bound in (4.41).

There exist indices $i, i' \in U$, $i \neq i'$, so that $p \in S_{[u]_m}i$ and $p' \in S_{[u]_m}i'$. We distinguish several cases according to the values of $i$ and $i'$.

Case 1. Suppose that $\{i, i'\} = \{(2, 1), (3, 2)\}$ or $\{i, i'\} = \{(2, 2), (3, 1)\}$. Then
\begin{equation}
\frac{1}{4} \cdot 4^{-m} \leq d(p, p') \leq \frac{1}{2} \cdot 4^{-m}.
\end{equation}
By (4.41) we see that (4.38) holds with $K_1 = 4$.

If $|t - t'| \geq \frac{1}{8} \cdot 4^{-m}$, then (4.39) holds with $K_2 = 4$ by (4.41) and (4.42). Suppose that $|t - t'| \leq \frac{1}{8} \cdot 4^{-m}$. Let $c_{u,m} = (a_{u,m} + b_{u,m})/2 = a_{u,m} + \frac{1}{2} \cdot 4^{-m}$. Then $|t - t'| = |t - c_{u,m}| + |c_{u,m} - t'|$ and
\[
D(A_t^v, A_{t'}^{v'}) \geq D(A_t^v, A_{c_{u,m}}^{v'}) - D(A_{c_{u,m}}^{v'}, A_{c_{u,m}}^{v'}) - D(A_{c_{u,m}}^{v'}, A_{t'}^{v'})
\geq D(A_{c_{u,m}}^{v'}, A_{c_{u,m}}^{v'}) - |c_{u,m} - t| - |c_{u,m} - t'|
\geq \frac{1}{4} \cdot 4^{-m} - |t - t'| > \frac{1}{8} \cdot 4^{-m},
\]
where we used Lemma 4.17 and Remark 4.40 in the second and third lines, respectively. Combining this with (4.41), we see that (4.39) holds with $K_2 = 4$.

We divide the remaining possibilities into two cases:
Case 2. \( \{i, i'\} \cap \{1, 4\} \neq \emptyset \) or \( \{i, i'\} = \{(2, 1), (3, 1)\} \) or \( \{i, i'\} = \{(2, 2), (3, 2)\} \).

Case 3. (a) \( \{i, i'\} = \{(3, 1), (3, 2)\} \) or (b) \( \{i, i'\} = \{(2, 1), (2, 2)\} \).

In case 2 we may assume that \( t < t' \) and choose

\[
t'' \in \left\{ a_{u,m} + \frac{1}{4} \cdot 4^{-m}, a_{u,m} + \frac{1}{2} \cdot 4^{-m}, a_{u,m} + \frac{3}{4} \cdot 4^{-m} \right\}
\]

so that \( t \leq t'' \leq t' \). In case 3(a) we choose \( t'' = a_{u,m} + \frac{1}{4} \cdot 4^{-m} \), while in case 3(b) we choose \( t'' = a_{u,m} + \frac{3}{4} \cdot 4^{-m} \). In all cases, for a suitable choice of \( p'' = [t'', v''_0] \), we have

\[
(t'', v) \sim (t'', v''_0) \sim (t'', v''_0)
\]

by (4.15) and Lemma 4.37(i). Thus (4.39) holds with \( K_1 = 1 \).

The proof of (4.39) splits according to the various cases.

Case 2. Since \( t < t'' < t' \), we obtain from (4.43) and (4.41) that \( d(p, p') = |t - t'| \leq D(A^v_r, A^v_{i'}) \). Thus (4.39) holds with \( K_2 = 1 \) in this case.

Case 3(a). In this case we have \( a_{u,m} + \frac{1}{2} \cdot 4^{-m} \leq t, t' \leq t'' = a_{u,m} + \frac{3}{4} \cdot 4^{-m} \). We may assume that \( i = (3, 1) \) and \( i' = (3, 2) \). By Lemma 4.36(i), \( A^v_{i'} \) contains a point \( x' \) such that

\[
a_{u,m} + \frac{1}{16} \cdot 4^{-m} \leq x' \leq \max\{a_{u,m} + \frac{1}{8} \cdot 4^{-m}, t' - \frac{1}{2} \cdot 4^{-m}\};
\]

Lemma 4.37(i) guarantees that \( A^v_r \subset [0, a_{u,m}] \cup [a_{u,m} + \frac{1}{4} \cdot 4^{-m}, t] \). Hence

\[
D(A^v_r, A^v_{i'}) \geq \min\{x' - a_{u,m}, a_{u,m} + \frac{1}{4} \cdot 4^{-m} - x'\} \geq \min\{\frac{1}{16} \cdot 4^{-m}, t'' - t'\} \geq \frac{1}{4}(t'' - t')
\]

since \( t'' - t' \leq \frac{1}{4} \cdot 4^{-m} \). Thus

\[
d(p, p') = 2t'' - t - t' \leq 2(t'' - t') + |t' - t| \leq 9D(A^v_r, A^v_{i'})
\]

by the first line of (4.43) and (4.41). We obtain (4.39) with \( K_2 = 9 \).

Case 3(b). In this case we have \( t'' = a_{u,m} + \frac{1}{4} \cdot 4^{-m} \leq t, t' \leq a_{u,m} + \frac{1}{2} \cdot 4^{-m} \). We may assume that \( i = (2, 1) \) and \( i' = (2, 2) \). If \( t - t'' \leq \frac{15}{16}(t' - t'') \), then

\[
D(A^v_r, A^v_{i'}) \geq |t - t'| \geq \frac{1}{16}(t' - t'').
\]

Otherwise, choose an integer \( k \geq 1 \) so that

\[
t'' + \frac{1}{4^k} \cdot 4^{-m} \leq t' \leq t'' + \frac{1}{4^{k-1}} \cdot 4^{-m}.
\]
By Lemma 4.36(ii), $A^\nu_t$ contains a point $x'$ such that
\[
a_{u,m} + \frac{1}{4^{k+1}} \cdot 4^{-m} \leq x' \leq t' - \left(\frac{1}{4} + \frac{3}{4^{k+1}}\right) 4^{-m}.
\]
Next, Lemma 4.37(ii) guarantees that $A^\nu_t \subset [0, a_{u,m}] \cup [t - \frac{1}{4} \cdot 4^{-m}, t]$. Hence
\[
D(A^\nu_t, A^\nu_{t'}) \geq \min\{x' - a_{u,m}, t - \frac{1}{4} \cdot 4^{-m} - x'\}
\geq \min\left\{\frac{1}{4^{k+1}} \cdot 4^{-m}, \frac{3}{4^{k+1}} \cdot 4^{-m} - |t - t'|\right\}
\geq \min\left\{\frac{1}{4^{k+1}} \cdot 4^{-m}, \frac{3}{4^{k+1}} \cdot 4^{-m} - \frac{1}{16} (t' - t'')\right\}
\geq \frac{1}{4^{k+1}} \cdot 4^{-m} \geq \frac{1}{16} (t' - t'').
\]
In either case, we find $D(A^\nu_t, A^\nu_{t'}) \geq \frac{1}{16} (t' - t'')$, whence
\[
d(p, p') = t + t' - 2t'' \leq 2(t' - t'') + |t' - t| \leq 33D(A^\nu_t, A^\nu_{t'})
\]
by the first line of (4.43) and (4.41). We obtain (4.39) with $K_2 = 33$. \qed

5. Questions and remarks

**Question 1.** Does $K([0,1])$ admit a bi-Lipschitz embedding in $L^1$?

The map $F : K([0,1]) \to L^1([0,1])$ given by $F(A) = \text{dist}(x, A)$ is a 1-Lipschitz embedding. Its inverse can be no better than $1/2$-Hölder, as can be seen by considering the distance from an arbitrary point $A \in K(X)$ to $B = [0,1]$. Is $F^{-1}$ $1/2$-Hölder continuous?

**Question 2.** Is there a constant $L < \infty$ so that every series-parallel graph of diameter one admits an $L$-bi-Lipschitz embedding in $K([0,1])$?

As previously observed, Gupta et al. [18] have recently shown that every series-parallel graph admits a 14-bi-Lipschitz embedding in $L^1$. Bi-Lipschitz embeddings of graphs and/or finite metric spaces in “good” targets play an increasingly important role in algorithmic problems in computer science and sorting problems (see [18] for a more complete list of applications). It would be interesting to know whether hyperspaces can play the role of the target in embedding results of this type.

**Question 3.** For which spaces $X$ does $K(X)$ admit a bi-Lipschitz embedding into a uniformly convex Banach space? into a finite dimensional Euclidean space?

Taking into account the principal theorems of this paper, it is natural to begin an attack on the first part of Question 3 by considering non-uniformly disconnected subsets of $[0,1]$, for example, the “fat” Cantor sets of [6]. When do the hyperspaces of such sets admit bi-Lipschitz embeddings in uniformly convex Banach spaces? For the second part of Question 3, see Remark 3.15.
REFERENCES


Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61822

E-mail address: tyson@math.uiuc.edu