SOBOLEV CLASSES OF BANACH SPACE-VALUED FUNCTIONS AND QUASICONFORMAL MAPPINGS

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ABSTRACT. We give a definition for the class of Sobolev functions from a metric measure space into a Banach space. We give various characterizations of Sobolev classes and study the absolute continuity in measure of Sobolev mappings in the "borderline case". We show under rather weak assumptions on the source space that quasisymmetric homeomorphisms belong to a Sobolev space of borderline degree; in particular, they are absolutely continuous. This leads to an analytic characterization of quasiconformal mappings between Ahlfors regular Loewner spaces akin to the classical Euclidean situation. As a consequence, we deduce that quasisymmetric maps respect the Cheeger differentials of Lipschitz functions on metric measure spaces with borderline Poincaré inequality.

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Contents

1. Introduction

Generalizations of the theory of Sobolev spaces have become a topic of increasing importance in recent years. Several authors (Hajłasz [17], Hajłasz-Koskela [18], Koskela-MacManus [32], Cheeger [9], Shanmugalingam [49]) have given definitions for Sobolev classes of real-valued functions on metric measure spaces. Ambrosio [2], Korevaar-Schoen [30], Reshetnyak [45] and Ranjbar-Motlagh [43] have studied Sobolev mappings from domains in Euclidean (or Riemannian) space into a complete metric space. See also [27] and the references therein. In this paper, we give a definition for the class of Sobolev functions from a metric measure space into a metric space that unifies many of the above approaches. As every metric space Y may isometrically be embedded in the Banach space $\ell^\infty(Y)$ of bounded functions on Y, it suffices to consider the case when the target is an arbitrary Banach space. There are several advantages in allowing for a Banach space target. First, we can conveniently invoke the vector-valued integration theory of Bochner and Pettis. Second, our function spaces have a linear structure; the Sobolev spaces to be defined are themselves Banach. Third, even when one is interested in the study of maps $F: X \to Y$ between (nonlinear) metric spaces, it is advantageous to regard Y as a subset of $\ell^\infty(Y)$, because certain discrete Lipschitz "convolution" approximations to F need not take values in Y but rather in $\ell^\infty(Y)$ (or in any given Banach space containing Y).

For all our considerations, the principal hypotheses are the validity of a Poincaré inequality as defined in [23], [24], and certain growth conditions on measure. We shall prove in Section 4 that the validity of a Poincaré inequality for mappings of a metric space is independent of the target Banach space.[1] In Section 6, we study embedding theorems and the Lipschitz approximation of Sobolev functions. The latter seems to be dependent on the geometric structure of the target; an approximation is possible if the values lie in a space that is an absolute Lipschitz retract. We also prove that pseudomonotone Sobolev mappings in the "borderline case" are absolutely continuous in measure (Theorem 7.2). This theorem generalizes results of Reshetnyak [44], Malý-Martio [36], and others from $\mathbb{R}^n$ to a general setting of metric spaces.

[1] Added in March 2001: we have recently applied the technique which we use in the proof of this result to answer a question of Jost [26, p. 12] regarding the validity of metric space-valued Poincaré inequalities for Dirichlet forms.

Our study was partially motivated by questions in the theory of quasiconformal mappings in metric spaces. In Euclidean space $\mathbb{R}^n$, one derives many important properties of quasiconformal homeomorphisms from the fact that these maps belong to the Sobolev space $W^{1,n}_{\mathrm{loc}}$ (by the *analytic definition* [14], [56, Theorem 34.6]). In the setting of general metric spaces, the most natural definition for quasiconformality is the *metric definition* (as in [56, 34.1], [24]). In a recent paper, Heinonen and Koskela [24] proved that a quasiconformal homeomorphism $F : X \to Y$ between two metric spaces belongs to the Sobolev spaces of Hajłasz and Korevaar-Schoen, provided that X and Y are Ahlfors regular of some dimension $Q > 1$ and admit a p-Poincaré inequality for some $p < Q$. In fact, in this case F belongs to the Sobolev space of order $Q + \epsilon$ for some $\epsilon > 0$ as in the celebrated theorem of Gehring in $\mathbb{R}^n$ [15]. Absolute continuity of F in measure follows easily from this higher integrability. In fact, it was demonstrated in [24] that in this case the pull-back measure under F is A_∞-related to the Hausdorff measure in the source; this implies a stronger, scale-invariant form of absolute continuity.

It was left open in [24] whether a quasiconformal homeomorphism $F : X \to Y$ is absolutely continuous in measure if a weaker Poincaré inequality holds on X and Y. The natural borderline case is best described by the validity of the so-called Q-Poincaré inequality, if X is Ahlfors Q-regular. We shall show here (Section 8) that in this case F indeed belongs to the Sobolev space of order Q, and is absolutely continuous in measure, provided the Hausdorff Q-measure in Y is locally finite. This follows from the aforementioned general result about pseudomonotone Sobolev mappings. The absolute continuity is a key to showing that the three classical definitions for quasiconformality – the analytic, metric, and geometric definition – can all equivalently be used in a very general context. That is the context of mappings between Ahlfors Q-regular spaces with Q-Poincaré inequality (or Q-Loewner spaces in the terminology of [23], [24]). We shall make all this precise in Section 9 below, where generalizations, historical comments, and applications can be found.

Finally, in Section 10, we shall apply the results of the previous sections and show that the fundamental commuting relation

$$dF^* = F^*d \tag{1.1}$$

remains valid for quasiconformal homeomorphisms F between Ahlfors Q-regular spaces with Q-Poincaré inequality. Here d is the Cheeger differential for Lipschitz functions introduced recently by Cheeger [9]. It was shown in [9] that (1.1) is true under the stronger hypothesis of p-Poincaré inequality for some $p < Q$; this hypothesis was tied up with the hypotheses needed for absolute continuity in [24] as explained above.

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Notation 1.2. We explain here the basic notation used throughout the paper. Much of the terminology of the introduction will be explained in the course of the paper.

We denote by $X = (X, d)$ an arbitrary metric space. For a ball $B = B(x, r)$ in X and for a number $\tau > 0$, we denote by τB the *dilated ball* $B(x, \tau r)$. Here B can be either open or closed; it is assumed that τB is of the same type.

The characteristic function of a subset $E \subset X$ will be denoted χ_E.

All measures μ on X will be assumed to be nontrivial, complete and Borel regular, and to assign finite and positive mass to each metric ball in X. For $1 \le p \le \infty$, we denote by

$L^p_{\mathrm{loc}}(X) = L^p_{\mathrm{loc}}(X, \mu)$ the class of all measurable functions $f : X \to \mathbb{R}$ such that each point $x \in X$ has a neighborhood U for which $f \in L^p(U)$.

We denote by V an arbitrary Banach space of positive dimension. The norm of an element $v \in V$ will be written $\|v\|$. We write V^* for the dual space of V, which we endow with the norm

$$\|v^*\| = \sup\{|\langle v^*, v\rangle| : v \in V, \|v\| \leq 1\}.$$

We shall make extensive use of the following version of the Hahn-Banach theorem: for each $v \in V$, $v \neq 0$, we can find $v^* \in V^*$ with $\|v^*\| = 1$ so that $\langle v^*, v\rangle = \|v\|$.

Finally, for a metric space Y, we denote by $\ell^\infty(Y)$ the Banach space consisting of all bounded real-valued functions on Y, which we endow with the supremum norm

$$\|f\|_{\ell^\infty(Y)} := \sup\{|f(y)| : y \in Y\}.$$

We recall that every metric space $Y = (Y, d)$ may be isometrically embedded in $\ell^\infty(Y)$. For example, fix a basepoint y_0 in Y and consider the mapping

$$y \mapsto f_y : Y \to \mathbb{R}, \qquad f_y(z) := d(y, z) - d(y_0, z). \tag{1.3}$$

2. Banach space-valued functions and vector-valued integration

In this section, we review the basic theory of measurability and (Bochner) integrability for vector-valued functions defined on a measure space. Throughout this section, we assume that $X = (X, \mu)$ is a σ-finite and complete measure space. (We do not assume until Definition 2.8 that X is a metric space.)

Theorem 2.1. *Let V be a Banach space and let $F : X \to V$. Then the following conditions are equivalent:*

1. *F is the pointwise a.e. limit of a sequence of simple functions;*
2. *F is essentially separably valued and $F^{-1}(U)$ is measurable for each open set $U \subset V$;*
3. *F is essentially separably valued and weakly measurable.*

We call a function $F : X \to V$ a *simple function* if there exists a finite collection of vectors $v_1, \ldots, v_n \in V$ together with pairwise disjoint measurable sets $E_1, \ldots, E_n \subset X$ so that

$$F = \sum_{i=1}^n v_i \chi_{E_i}.$$

We call F *essentially separably valued* if there exists $Z \subset X$ with $\mu(Z) = 0$ so that $F(X \setminus Z)$ is a (norm) separable subset of V.

We call F *weakly measurable* if the scalar function $\langle v^*, F\rangle : X \to \mathbb{R}$ is measurable for each $v^* \in V^*$.

Definition 2.2. We say that F is *measurable* if one (and hence each) of the three conditions in Theorem 2.1 holds.

Remark 2.3. The condition in (2) that F be essentially separably valued is automatically satisfied if V has a dense subset whose cardinal is an *Ulam number*, see [11, 2.1.6, 2.3.8].

The equivalence of the above three definitions of measurability is well known. See [11, 2.3.8] for definition (2) and [10, Chapter II] for definitions (1) and (3). The equivalence of (1) and (3) is the content of the Pettis Measurability Theorem, see [10, Chapter II, §1, Theorem 2]. The implication (2) ⇒ (3) is trivial, and the implication (1) ⇒ (2) is an easy exercise.

Definition 2.4. Let $F : X \to V$ be a measurable function. We say that F is *(Bochner) integrable* if there exists a sequence (F_ν) of simple functions with $\|F_\nu\| \in L^1(X)$ so that $\int_X \|F(x) - F_\nu(x)\|\, d\mu(x) \to 0$. In this case, for each measurable set $E \subset X$, we define the *Bochner integral of F over E* to be the value

$$\int_E F(x)\, d\mu(x) = \lim_{\nu\to\infty} \int_E F_\nu(x)\, d\mu(x),$$

where (F_ν) is a sequence as above and $\int_E F_\nu\, d\mu$ is defined in the obvious way.

One easily verifies that this limit exists and is independent of the choice of defining sequence. We denote by $L^1(X, \mu : V) = L^1(X : V)$ the collection of all Bochner integrable functions from X to V. As usual, we identify two functions $F, G \in L^1(X : V)$ for which $F = G$ a.e.

We introduce a norm in the space $L^1(X : V)$ as follows. First, for a simple function $F = \sum_{i=1}^n v_i \chi_{E_i}$ (with $E_1, \ldots, E_n$ disjoint sets of finite measure in X), we set

$$\|F\|_1 = \int_X \|F(x)\|\, d\mu(x) = \sum_{j=1}^n \mu(E_j)\|v_j\|. \tag{2.5}$$

We extend this norm to all of $L^1(X : V)$ by continuity. With the help of Theorem 2.1, we can easily establish the following result (see [10, Chapter II, §2, Theorem 2]):

Proposition 2.6. *A measurable function $F : X \to V$ is Bochner integrable if and only if $\|F\| \in L^1(X)$.*

For $1 \le p < \infty$, we define $L^p(X : V)$ to be the collection of (equivalence classes of) measurable maps $F : X \to V$ for which $\|F\| \in L^p(X)$, endowed with the norm $\|F\|_p = (\int_X \|F(x)\|^p\, d\mu(x))^{1/p}$. Proposition 2.6 shows that this agrees with the earlier definition in the case $p = 1$. We let $L^\infty(X : V)$ denote the collection of (equivalence classes of) essentially bounded measurable maps from X to V, endowed with the norm $\|F\|_\infty = \operatorname{ess\,sup}_{x\in X}\|F(x)\|$. For any $1 \le p \le \infty$, the normed space $L^p(X : V)$ is a Banach space.

We let $L^p_{\mathrm{loc}}(X : V)$ denote the collection of measurable maps $F : X \to V$ for which $\|F\| \in L^p_{\mathrm{loc}}(X)$. Maps in $L^1_{\mathrm{loc}}(X : V)$ are said to be *locally (Bochner) integrable.*

For $F \in L^1(X : V)$, and for $E \subset X$ so that $\mu(E) > 0$, we define the *mean value* of F over E to be the vector

$$F_E = -\!\!\!\!\!\!\int_E F(x)\, d\mu(x) := \frac{1}{\mu(E)} \int_E F(x)\, d\mu(x). \tag{2.7}$$

The vector F_E lies in the *closed convex hull* $\overline{co}(F(E))$ of $F(E)$. Recall that, for an arbitrary subset $A \subset V$, $\overline{co}(A)$ consists of those $w \in V$ for which

$$\inf_{v\in A} \langle v^*, v\rangle \le \langle v^*, w\rangle \le \sup_{v\in A} \langle v^*, v\rangle$$

for all $v^* \in V^*$.

Remark 2.8. We assume henceforth that $X = (X, d, \mu)$ is a *metric measure space* as defined in 1.2; recall that this includes the assumption that the measure of each ball in X is finite and positive. We call μ a *Vitali measure* if the conclusion of Vitali's covering theorem holds: whenever $\mathcal{B}$ is a covering of a set $A \subset X$ by closed balls with $\inf\{r : B(x, r) \in \mathcal{B}\} = 0$ for each $x \in A$, then there exist disjoint balls $B_1, B_2, \ldots$ in $\mathcal{B}$ with $\mu(A \setminus \bigcup_i B_i) = 0$.

For example, if μ is *doubling*, which means that there is a constant $C_\mu \geq 1$ such that

$$\mu(B(x,2r)) \leq C_\mu \, \mu(B(x,r)) \tag{2.9}$$

for each $x \in X$ and $0 < r < \operatorname{diam} X$, then μ is a Vitali measure, see [11, 2.8]. Furthermore, every Radon measure on $\mathbb{R}^n$ is a Vitali measure, see, e.g., [38, Theorem 2.8].

If μ is a doubling measure on X, then we call the triple $X = (X, d, \mu)$ a *doubling metric measure space.* Note that every doubling metric measure space X is separable, whence continuous functions from X into a Banach space are measurable by parts (2) or (3) of Theorem 2.1.

Recalling that we assume that measurable mappings are essentially separably valued, the following proposition can be proved as in [11, 2.9.9].

Proposition 2.10. *Let μ be a Vitali measure on the metric space and let V be a Banach space. If $F : X \to V$ is locally Bochner integrable, then almost every point $x \in X$ is a Lebesgue point of F, i.e.*

$$\lim_{r \to 0} \fint_{B(x,r)} \|F(y) - F(x)\| \, d\mu(y) = 0. \tag{2.11}$$

In particular, at each Lebesgue point x, the vectors $F_{B(x,r)}$ converge to $F(x)$ as $r \to 0$.

Note that by the Hahn-Banach theorem, $\|\int_A G(y)\, d\mu(y)\| \leq \int_A \|G(y)\| \, d\mu(y)$ whenever $G : X \to V$ is a measurable function and $A \subset X$ is a measurable set.

3. The Sobolev space $N^{1,p}(X : V)$

Let $X = (X, d, \mu)$ be a metric measure space as stipulated in 1.2 and let $1 \leq p < \infty$.

We first recall the definition for a Sobolev space of real-valued functions given in [49].

Definition 3.1. A measurable function $f : X \to \mathbb{R}$ is said to be in the *(Newtonian) Sobolev space* $N^{1,p}(X)$ if $f \in L^p(X)$ and if there exists a Borel function $\rho : X \to [0, \infty]$ so that $\rho \in L^p(X)$ and

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma \rho \, ds \tag{3.2}$$

for p-a.e. rectifiable curve $\gamma : [a, b] \to X$.

In (3.2), $\int_\gamma \rho \, ds$ denotes the line integral of ρ along γ. A collection Γ of locally rectifiable curves in X is called *p-exceptional* if there exists a nonnegative Borel function $\rho \in L^p(X)$ with $\int_\gamma \rho \, ds = \infty$ for all $\gamma \in \Gamma$. We say that a property of curves holds for *p-almost every* *(p-a.e.)* curve if the collection of curves for which the property fails to hold is p-exceptional. Alternatively, Γ is p-exceptional if its *p-modulus* $\operatorname{mod}_p \Gamma$ is equal to zero, where, for each curve family Γ we define

$$\operatorname{mod}_p \Gamma = \inf \int_X \rho^p \, d\mu \, , \tag{3.3}$$

the infimum being taken over all Borel functions $\rho : X \to [0, \infty]$ that are *admissible for* Γ, that is, $\int_\gamma \rho \, ds \geq 1$ for each locally rectifiable $\gamma \in \Gamma$. See [56, Chapter 1] for a thorough discussion on line integrals and modulus on $\mathbb{R}^n$; the discussion in [56] generalizes to metric measure spaces in a straightforward manner. (See also [24], [49].)

A function ρ satisfying (3.2) for p-a.e. rectifiable curve γ in X is called a *p-weak upper gradient* of f. If ρ satisfies (3.2) for every rectifiable curve γ, we call it an *upper gradient* of f. It easily follows from the definition for a p-exceptional curve family that if a function possesses a p-integrable p-weak upper gradient ρ, it also possesses a p-integrable upper gradient in each L^p neighborhood of ρ. A standard convexity argument from functional analysis (Mazur's lemma) together with a lemma of Fuglede (Lemma 3.4 below) implies that, if $1 < p < \infty$, there exists a *minimal p-weak upper gradient* $\rho_f \in L^p(X)$, unique up to modification on sets of measure zero. The minimality of ρ_f means that $\rho_f \leq \rho$ a.e. for every p-weak upper gradient ρ of f.

For the record, we state the following lemma due to Fuglede [12]:

Lemma 3.4. *If a sequence of Borel functions (g_n) converges in $L^p(X)$, then the limit function has a Borel representative g, and there is a subsequence (g_{n_k}) such that*

$$\int_\gamma |g_{n_k} - g|\, ds \to 0$$

for p-a.e. curve γ in X.

Remark 3.5. The above lemma is usually stated with the assumption that the sequence (g_n) converges in $L^p(X)$ to a Borel function. In our situation, it is a standard consequence of Lusin's theorem that the L^p limit can be corrected in a set of measure zero so as to become Borel [11, 2.3.6].

Next, we define the *$N^{1,p}$-norm* of the function f to be

$$\|f\|_{1,p} = \|f\|_p + \inf_\rho \|\rho\|_p\,, \tag{3.6}$$

where the infimum is taken over all p-weak upper gradients (equivalently, for all upper gradients) ρ of f. If $1 < p < \infty$, then we have

$$\|f\|_{1,p} = \|f\|_p + \|\rho_f\|_p. \tag{3.7}$$

To be more precise, definition (3.6) only gives a seminorm in general. In [49], the elements of $N^{1,p}(X)$ are defined to be equivalence classes of functions, where $f_1 \sim f_2$ if and only if $\|f_1 - f_2\|_{1,p} = 0$. We shall adopt the same convention. With the usual abuse of terminology, the elements in $N^{1,p}(X)$ will still be referred to as functions. One should notice, however, that each representative f of an element in $N^{1,p}(X)$ has well defined values *up to a set of p-capacity zero*. Moreover, in many cases, e.g. for domains in $\mathbb{R}^n$, each representative is automatically *p-quasicontinuous*. It is important to notice this difference between the Newtonian Sobolev space and the standard Sobolev space $W^{1,p}$; it is true that the members of $W^{1,p}$ also can be represented by quasicontinuous functions, but that is not obvious from the definition and non-quasicontinuous representatives of $W^{1,p}$ functions exist in each equivalence class.

The normed space $N^{1,p}(X)$ is always a Banach space and agrees with the classical Sobolev space $W^{1,p}(\Omega)$ whenever $X = \Omega$ is a domain in $\mathbb{R}^n$ (with the Euclidean metric and Lebesgue measure); in this case, the minimal p-weak upper gradient ρ_f of a Sobolev function f agrees with the norm of its classical gradient $|\nabla f|$ almost everywhere. For these facts and terminology, see [49].

Remarks 3.8. The above definition as well as the term *Newtonian* for the space $N^{1,p}(X)$ is due to Shanmugalingam [49]. In [9], Cheeger defined a Sobolev space $H^{1,p}(X)$ in an arbitrary metric measure space X by using upper gradients differently. For $p > 1$, these two spaces are isometrically isomorphic by [48, Section 2.3]. Cheeger [9, Theorem 4.48] proved the important theorem that for $p > 1$ the spaces $H^{1,p}(X)$ are reflexive provided that μ is doubling and X supports the p-Poincaré inequality (see Section 4). It is not known whether the spaces $N^{1,p}(X)$ (for $p > 1$) are reflexive in general.

In [45], Reshetnyak introduces a notion of Sobolev classes for mappings from a bounded Euclidean domain Ω into an arbitrary complete metric space $Y = (Y, d)$. Earlier, Korevaar and Schoen [30] gave a different but equivalent definition with more general Riemannian source space, see Section 5. The definition of Korevaar and Schoen readily extends to the case where the domain is a general metric measure space; see [24, 9.6], [43].

Reshetnyak's definition is based on the elementary observation that postcomposition by a Lipschitz function preserves membership in the class $W^{1,p}(\Omega)$. The *Reshetnyak-Sobolev* class $W^{1,p}(\Omega : Y)$ consists of those functions $f : \Omega \to Y$ with the following property: for every 1-Lipschitz function $\varphi : Y \to \mathbb{R}$, the map $\varphi \circ f : \Omega \to \mathbb{R}$ is in the class $W^{1,p}(\Omega)$; furthermore, one requires that there exists a real-valued function $w \in L^p(\Omega)$, that does not depend on φ, so that the inequality $|\nabla(\varphi \circ f)(x)| \leq w(x)$ holds for a.e. $x \in \Omega$. In fact, by [45, Theorem 5.1], it suffices to require this for the functions φ which are of the form $\varphi(y) = d_z(y) = d(y, z)$, $z \in Y$.

We next combine the approaches of Shanmugalingam and Reshetnyak to give a definition of Sobolev mappings from a metric measure space into a metric space. We shall only consider the case when the target is a Banach space V; as explained in the introduction, this entails no loss of generality. Now there are several possible definitions of Sobolev functions in the sense of Reshetnyak. In addition to postcomposing with Lipschitz functions on V, or functions of the form d_z, $z \in V$, we may also use the linear functionals $v^* \in V^*$. We shall see below that all of these approaches give rise to the same class of mappings.

We shall generally consider mappings whose domains have finite mass.

Definition 3.9. Let V be a Banach space and let $X = (X, d, \mu)$ be a metric measure space with finite total measure. We say that a measurable map $F : X \to V$ is in the *(Reshetnyak-Newtonian) Sobolev class* $N^{1,p}(X : V)$, if $F \in L^p(X : V)$ and if there exists a Borel function $\rho : X \to [0, \infty]$ so that $\rho \in L^p(X)$ and that

$$\|F(\gamma(a)) - F(\gamma(b))\| \leq \int_\gamma \rho \, ds \tag{3.10}$$

for p-a.e. rectifiable curve $\gamma : [a, b] \to X$. Each such function ρ is called a *p-weak V-upper gradient* of F. If ρ satisfies (3.10) for all rectifiable curves γ, then ρ is called a *V-upper gradient* of F. It is clear that every function in $N^{1,p}(X : V)$ has a V-upper gradient in $L^p(X)$.

When $\mu(X)$ is not necessarily finite, we define the *local Sobolev class* $N^{1,p}_{\mathrm{loc}}(X : V)$ to consist of those functions $F \in L^p_{\mathrm{loc}}(X : V)$ that have a (p-weak) V-upper gradient $\rho \in L^p_{\mathrm{loc}}(X)$.

Sometimes we wish to consider mappings from X into an arbitrary metric space Y. In this case we employ the notation

$$N^{1,p}(X : Y) = \{F \in N^{1,p}(X : \ell^\infty(Y)) : F(x) \in Y \text{ for } p\text{-quasi-every } x \in X\}, \tag{3.11}$$

and similarly for the local space $N^{1,p}_{\mathrm{loc}}(X : Y)$.[2] Here the phrase *"for p-quasi-every x"* means that the set of exceptional points has p-capacity zero; recall that functions in $N^{1,p}(X : V)$

[2]Note that strictly speaking the definition of $N^{1,p}(X : Y)$ in (3.11) depends on the choice of a basepoint y_0 in Y — at least if we use the embedding of Y in $\ell^\infty(Y)$ given in (1.3). It would thus be more precise to define

are well defined up to sets of p-capacity zero (see, e.g., [49]). Recall also that $\ell^\infty(Y)$ denotes the Banach space of bounded functions on Y.

We observe that $N^{1,p}(X:V)$ is a linear space that can be (semi-)normed by

$$\|F\|_{1,p} = \|F\|_p + \inf_\rho \|\rho\|_p\,, \tag{3.12}$$

where the infimum is taken over all p-weak V-upper gradients (equivalently, over all V-upper gradients) ρ of F. As in the case of real-valued functions, we assume from now on that *the elements of $N^{1,p}(X:V)$ are equivalence classes of maps, where $F_1 \sim F_2$ if and only if* $\|F_1 - F_2\|_{1,p} = 0$. Thus we get that $\|\cdot\|_{1,p}$ as defined in (3.12) is a norm. The arguments in [49] can now be adopted almost verbatim to reach the following conclusion:

Theorem 3.13. *The normed space $(N^{1,p}(X:V), \|\cdot\|_{1,p})$ is a Banach space. When $p>1$, every function $F \in N^{1,p}(X:V)$ has a minimal p-weak V-upper gradient ρ_F in $L^p(X)$.*

Remark 3.14. We have chosen to use the traditional terminology "Sobolev class" here while retaining the notation $N^{1,p}(X:V)$ so as to emphasize the special nature of its elements. Namely, it follows from the arguments in [49], that functions in $N^{1,p}(X:V)$ are unambiguously defined up to a set of p-capacity zero. Here p-capacity is an intrinsic (outer) measure on X, independent of the target space V. In particular, one should keep in mind that we are not, in general, free to change the values of the elements in $N^{1,p}(X:V)$ on sets of measure zero. We also hope that the notation will enable the readers to distinguish between $N^{1,p}(X:V)$ and the other burgeoning, possibly different Sobolev classes of mappings between metric spaces.

Example 3.15. Let $F: X \to V$ be locally Lipschitz, i.e. F is Lipschitz in a neighborhood of each point in X. Then $F \in N^{1,p}_{\mathrm{loc}}(X:V)$ for all $1 \le p < \infty$. For future reference, we note here that the locally bounded function $\operatorname{lip} F : X \to \mathbb{R}$, defined by

$$\operatorname{lip} F(x) = \liminf_{r\to 0} r^{-1} \sup_{\substack{y\\ d(x,y)\le r}} \|F(x) - F(y)\|$$

is a V-upper gradient of f. *A fortiori*, the function $\operatorname{Lip} F : X \to \mathbb{R}$, defined by

$$\operatorname{Lip} F(x) = \limsup_{r\to 0} r^{-1} \sup_{\substack{y\\ d(x,y)\le r}} \|F(x) - F(y)\|$$

is also a V-upper gradient of f.

Let us recall the standard argument which shows that $\operatorname{lip} F$ is a V-upper gradient of F (compare [46, Lemma 1.20]). Suppose that $\gamma : [a,b] \to X$ is a rectifiable curve (parameterized by arc length) with $\gamma(a) = x$ and $\gamma(b) = y$. Then F is Lipschitz on the image of γ. We may clearly assume that $F(x) \ne F(y)$. Choose $v^* \in V^*$ with $\|v^*\| = 1$ so that $\|F(x) - F(y)\| = \langle v^*, F(x) - F(y)\rangle$. Then the mapping

$$t \mapsto \langle v^*, (F\circ\gamma)(t)\rangle \tag{3.16}$$

from $[a,b]$ to $\mathbb{R}$ is Lipschitz and hence differentiable at almost every $t \in [a,b]$; moreover,

$$\langle v^*, (F\circ\gamma)(b)\rangle - \langle v^*, (F\circ\gamma)(a)\rangle = \int_a^b \langle v^*, (F\circ\gamma)\rangle'(t)\, dt$$

---

a notion of Sobolev function from a metric space X to a metric space with basepoint (Y, y_0). Throughout this paper, we will assume that a fixed choice of basepoint has been made in Y.

and so

$$\|F(x) - F(y)\| \le \int_a^b |\langle v^*, (F\circ\gamma)\rangle'(t)|\, dt \le \int_a^b (\operatorname{lip} F)(\gamma(t))\, dt = \int_\gamma \operatorname{lip} F\, ds$$

since $|\langle v^*, (F\circ\gamma)\rangle'(t)| \le (\operatorname{lip} F)(\gamma(t))$ at every point $t \in [a,b]$ where the map in (3.16) is differentiable.

We find it convenient to assume that the total measure of X is finite in Theorem 3.17. Otherwise, for example, nonzero constants are not in L^p, and the theorem would not be true as stated. The obvious analog of this theorem for the local Sobolev class $N^{1,p}_{\mathrm{loc}}$ holds irrespective of the measure of X.

Theorem 3.17. *Let $X = (X,\mu)$ be a metric measure space of finite total measure and let V be a Banach space. Then the following four conditions are equivalent for a function $F \in L^p(X:V)$:*

(1) *$F \in N^{1,p}(X:V)$;*

(2) *for each 1-Lipschitz function $\varphi : V \to \mathbb{R}$, the map $\varphi\circ F : X \to \mathbb{R}$ is in $N^{1,p}(X)$, and there exists $\rho \in L^p(X)$ that is an upper gradient of $\varphi \circ F$ for all such φ;*

(3) *for each $v^* \in V^*$ with $\|v^*\| \le 1$, the map $\langle v^*, F\rangle : X \to \mathbb{R}$ is in $N^{1,p}(X)$, and there exists $\rho \in L^p(X)$ that is an upper gradient of $\langle v^*, F\rangle$ for all such v^*;*

(4) *for each $z \in F(X)$, the map $d_zF : X \to \mathbb{R}$ defined by $d_zF(x) = \|F(x) - z\|$ is in $N^{1,p}(X)$, and there exists $\rho \in L^p(X)$ that is an upper gradient of d_zF for all such z.*

If we assume a priori *that $F(X)$ is a separable subset of V, and that $1 < p < \infty$, then the above are further equivalent to*

(2') *for each 1-Lipschitz function $\varphi : V \to \mathbb{R}$, the map $\varphi\circ F : X \to \mathbb{R}$ is in $N^{1,p}(X)$, and there exists $\tilde\rho \in L^p(X)$ so that $\rho_{\varphi\circ F} \le \tilde\rho$ a.e. for all such φ;*

(3') *for each $v^* \in V^*$ with $\|v^*\| \le 1$, the map $\langle v^*, F\rangle : X \to \mathbb{R}$ is in $N^{1,p}(X)$, and there exists $\tilde\rho \in L^p(X)$ so that $\rho_{\langle v^*, F\rangle} \le \tilde\rho$ a.e. for all such v^*;*

(4') *for each $z \in F(X)$, the map $d_zF : X \to \mathbb{R}$ defined by $d_zF(x) = \|F(x) - z\|$ is in $N^{1,p}(X)$, and there exists $\tilde\rho \in L^p(X)$ so that $\rho_{d_zF} \le \tilde\rho$ a.e. for all such z.*

In this latter case, there exists a countable set of linear functionals (v_n^*) with $\|v_n^*\| \le 1$ and a countable set of points (z_n) in $F(X)$ so that the following estimates for the minimal p-weak V-upper gradient ρ_F hold true:

$$\rho_F(x) \le \sup_n \rho_{\langle v_n^*, F\rangle}(x) \tag{3.18}$$

$$\rho_F(x) \le \sup_n \rho_{d_{z_n}F}(x) \tag{3.19}$$

for a.e. x in X.

Remarks 3.20. (1) In the event that X is separable and F is p-quasicontinuous, there exists a set $Z \subset X$ of zero p-capacity for which $F(X\setminus Z)$ is separable in V and it is straightforward to verify that this condition suffices for the second half of Theorem 3.17. Recall that separability of X is guaranteed by the doubling condition (2.9) on the measure μ.

(2) As previously observed, there is a subtle distinction between $N^{1,p}(X)$ and the classical Sobolev space $W^{1,p}(X)$ in the case when the source space is Riemannian: $N^{1,p}$ functions are

automatically quasicontinuous, while $W^{1,p}$ functions need not be (although every $W^{1,p}$ function has a quasicontinuous representative). Consequently, in the case when X is Riemannian, it is not clear whether the analog of Theorem 3.17 with $N^{1,p}(X)$ replaced by $W^{1,p}(X)$ must necessarily hold. In applications of these results in Section 5, it is this latter formulation which will be of use. To simplify the exposition, we postpone the statement and proof of this alternate formulation of Theorem 3.17 until it is needed. See Proposition 5.4.

In the following corollary, for a measure space $X = (X, \mu)$ of finite total measure, the class $L^p(X : Y)$ denotes the collection of measurable functions $F : X \to Y$ for which $d_z F \in L^p(X)$, where $z \in Y$ is a fixed basepoint; the definition of $L^p(X : Y)$ is clearly independent of the choice of basepoint. Equivalently, $L^p(X : Y)$ may be identified with the set of functions $F \in L^p(X : \ell^\infty(Y))$ for which $F(x) \in Y$ for a.e. $x \in X$. It is clear that this definition agrees with the definition given in Section 2 in the case when $Y = V$ is a Banach space.

Corollary 3.21. *Let $X = (X, \mu)$ be a metric measure space of finite total measure and let Y be an arbitrary metric space. Let $F \in L^p(X : Y)$. If we replace V by Y throughout the statement of Theorem 3.17, then conditions (1), (2) and (4) of that theorem are still equivalent; moreover, if we assume in addition that $F(X)$ is separable in Y and $p > 1$, then (2') and (4') are also equivalent and (3.19) continues to hold.*

Proof of Theorem 3.17. (1) $\Rightarrow$ (2). Let $\rho \in L^p(X)$ be an upper gradient of F, and let $\varphi : V \to \mathbb{R}$ be 1-Lipschitz. If γ is a rectifiable curve in X with end points x and y, then

$$|\varphi \circ F(x) - \varphi \circ F(y)| \le \|F(x) - F(y)\| \le \int_\gamma \rho\, ds.$$

On the other hand, because X has finite mass, we have that

$$\|\varphi \circ F\|_p \le \|F\|_p + \mu(X)^{1/p} |\varphi(0)| < \infty.$$

Thus $\varphi \circ F$ belongs to $N^{1,p}(X : V)$ and ρ is an upper gradient of $\varphi \circ F$, independent of φ.

(2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4). These two implications are trivial, since the mappings $d_z : V \to \mathbb{R}$, $z \in V$, given by $d_z(v) = \|v - z\|$ and the mappings $\langle v^*, \cdot \rangle : V \to \mathbb{R}$, $v^* \in V^*$, $\|v^*\| \le 1$, are 1-Lipschitz maps.

(3) $\Rightarrow$ (1), (4) $\Rightarrow$ (1). The proofs for these two implications are similar; we only show the first implication. Thus, let $F \in L^p(X : V)$ satisfy (3), and let $\rho \in L^p(X)$ be as in (3). Let γ be a rectifiable curve in X with end points x and y; we wish to show that $\|F(x) - F(y)\| \le \int_\gamma \rho\, ds$. If $F(x) = F(y)$ this is trivial. Otherwise, let $v^* \in V^*$ satisfy

$$\langle v^*, F(x) - F(y) \rangle = \|F(x) - F(y)\|$$

with $\|v^*\| \le 1$. Then, because ρ is an upper gradient of $\langle v^*, F \rangle$, we have that

$$\|F(x) - F(y)\| = \langle v^*, F(x) - F(y) \rangle \le \int_\gamma \rho\, ds$$

as required.

Assume now that $F(X)$ is a separable subset of V and that $1 < p < \infty$. Clearly, (2) implies (2') which in turn implies (3') and (4').

(3') $\Rightarrow$ (1). Let $(v_n) \subset V$ be a countable dense set in the difference set $F(X) - F(X) \subset V$. Without loss of generality we assume that $v_n \neq 0$ for each n. Denote by (v_n^*) a countable subset of V^* so that $\langle v_n^*, v_n \rangle = \|v_n\|$ and that $\|v_n^*\| = 1$. By assumption, inequality (3.10) holds for $\rho = \rho_{\langle v_n^*, F \rangle}$, for all curves γ outside a p-exceptional family Γ_n. Because $\Gamma_1 \cup \Gamma_2 \cup \cdots$ is also p-exceptional, it follows from Lemma 3.23 below that there is a family Γ of rectifiable curves in X with $\mathrm{mod}_p\{\gamma : \gamma \notin \Gamma\} = 0$ so that

$$|\langle v_n^*, F \rangle(\gamma(a)) - \langle v_n^*, F \rangle(\gamma(b))| \leq \int_\gamma \tilde{\rho}\, ds$$

for all $n = 1, 2, \ldots$ and $\gamma \in \Gamma$, $\gamma : [a, b] \to X$, where $\tilde{\rho}$ is as in (3').

Next, let $\gamma \in \Gamma$ be a curve with end points x and y. Because the sequence (v_n) is dense in $F(X) - F(X)$, we can find a subsequence (v_{n_j}) converging to $F(x) - F(y)$. Thus, with an obvious notation $v_{n_j}^*$, we find that

$$\begin{aligned} \|F(x) - F(y)\| &= \lim_{j \to \infty} \|v_{n_j}\| \\ &\leq \limsup_{j \to \infty} \big(|\langle v_{n_j}^*, v_{n_j} - (F(x) - F(y)) \rangle| + |\langle v_{n_j}^*, F(x) - F(y) \rangle| \big) \\ &\leq \limsup_{j \to \infty} \|v_{n_j} - (F(x) - F(y))\| + \int_\gamma \tilde{\rho}\, ds = \int_\gamma \tilde{\rho}\, ds\,. \end{aligned} \tag{3.22}$$

This proves that F belongs to $N^{1,p}(X : V)$.

The proof for the implication (4') $\Rightarrow$ (1) is similar, using the fact that

$$|d_{v_{n_j}} F(x) - d_{v_{n_j}} F(y)| \to \|F(x) - F(y)\|$$

for a sequence (v_{n_j}) in $F(X)$ converging to $F(x)$.

Finally, let us prove (3.18) and (3.19). Let

$$\rho_F^*(x) := \sup_n \rho_{\langle v_n^*, F \rangle}(x)$$

for a.e. $x \in X$, where the functionals v_n^* are chosen as in the above proof that (3') implies (1). Then that proof remains true if we replace $\tilde{\rho}$ with ρ_F^*. Although we do not know that $\rho_F^* \in L^p(X)$, we may still conclude from (3.22) that ρ_F^* is a p-weak V-upper gradient of F and hence $\rho_F(x) \leq \rho_F^*(x)$ for a.e. $x \in X$. The statement involving the functions d_z, $z \in F(X)$ is similar.

The proof for Theorem 3.17 is now complete. □

Lemma 3.23. *Let g, h be nonnegative Borel functions on a metric measure space X with $g \leq h$ a.e. Let $p \geq 1$. Then $\int_\gamma g\, ds \leq \int_\gamma h\, ds$ for p-a.e. curve γ.*

Proof. Denote by Γ the collection of curves γ for which $\int_\gamma g\, ds > \int_\gamma h\, ds$. It is clear that Γ is p-exceptional, because the Borel function $\rho = \infty \cdot \max\{g - h, 0\}$ both satisfies $\int_\gamma \rho\, ds = \infty$ for $\gamma \in \Gamma$ and belongs to $L^p(X)$. The lemma follows. □

Remark 3.24. The Sobolev space $W^{1,p}(M : N)$ for maps $F : M \to N$ between two Riemannian manifolds is customarily defined via an isometric embedding of N in some Euclidean space $\mathbb{R}^\nu$. Namely, $F \in W^{1,p}(M : N)$ if and only if $F(x) \in N$ for a.e. $x \in M$ and the component functions of $F : M \to \mathbb{R}^\nu$ belong to the standard Sobolev space $W^{1,p}$. Local spaces

$W^{1,p}_{\mathrm{loc}}(M:N)$ are defined similarly. Although the embedding $i : N \hookrightarrow \mathbb{R}^\nu$ is not isometric in the same sense as the embedding $N \hookrightarrow \ell^\infty(N)$ in (1.3) is (one uses the intrinsic metric on $i(N) \subset \mathbb{R}^\nu$), it is nevertheless straightforward to verify that a map $F : M \to N$ belongs to $N^{1,p}_{\mathrm{loc}}(M:N)$ if and only if it belongs to $W^{1,p}_{\mathrm{loc}}(M:N)$, and that $\inf_\rho \|\rho\|_p = \|\nabla F\|_p$, where the infimum is taken over all upper gradients ρ of F. In particular, if M and N are compact (possibly with boundary), then $N^{1,p}(M:N) = W^{1,p}(M:N)$.

4. Poincaré inequalities for Banach space-valued maps

In this section, we discuss Poincaré inequalities for Banach space-valued maps. The principal result (Theorem 4.3) is that the validity of a Poincaré inequality is independent of the target space. The discussion here uses upper gradients as defined in the previous section but is independent of the notion of a Sobolev space. For a comprehensive theory of Poincaré inequalities for real valued functions in the setting of general metric spaces, see [18].

In what follows, $X = (X, d, \mu)$ denotes a metric measure space as in 1.2, $1 \le p < \infty$, and V is a Banach space. We denote by $\mathcal{L}ip_{\mathrm{loc}}(X:V)$ the class of locally Lipschitz maps from X to V, by $\mathcal{C}(X:V)$ the class of continuous maps from X to V, and by $\mathcal{M}(X:V)$ the class of measurable maps from X to V. We also use the self-explanatory notation $\mathcal{L}ip_{\mathrm{loc}}(A:V)$, $\mathcal{C}(A:V)$, and $\mathcal{M}(A:V)$ for subsets $A \subset X$.

Definition 4.1. Let B be an open ball in X, $\sigma \ge 1$, $F \in L^1(\sigma B : V)$, and $\rho : \sigma B \to [0,\infty]$ Borel measurable. (Recall the notation from 1.2.) We say that the pair of functions F and ρ satisfies the *p-Poincaré inequality* in σB if there exists a constant $C_P > 0$ so that

$$\fint_B \|F - F_B\|\,d\mu \le C_P(\operatorname{diam} B)\left(\fint_{\sigma B} \rho^p\,d\mu\right)^{1/p}. \tag{4.2}$$

Here F_B denotes the mean value of the Banach space-valued function F on the ball B, defined in (2.7).

Next, let $\mathcal{S}$ be one of the classes $\mathcal{L}ip_{\mathrm{loc}}$, $\mathcal{C}$ or $\mathcal{M}$. If condition (4.2) holds for all balls B in X, for all L^1-functions $F \in \mathcal{S}(\sigma B : V)$, and for all p-weak V-upper gradients ρ of F in the ball σB, with constants C_P and σ independent of B, F, and ρ, we say that the pair (X,V) *supports the p-Poincaré inequality for the class* $\mathcal{S}$. When $V = \mathbb{R}$, we just say that X supports the p-Poincaré inequality for the class $\mathcal{S}$; if $\mathcal{S} = \mathcal{M}$, we simply say that X supports the p-Poincaré inequality.

Note the terminological difference with [24], where the term *weak* $(1,p)$*-Poincaré inequality* was used. Note also that in [24], Poincaré inequalities were formulated for upper gradients rather than weak upper gradients; the two notions are easily seen to be equivalent. For a relationship between the different classes $\mathcal{S}$ in this context, see Remark 4.4 below.

For the next theorem, recall the definition for a doubling metric measure space from 2.8.

Theorem 4.3. *Assume that $X = (X, d, \mu)$ is a doubling metric measure space and let $\mathcal{S}$ be one of the classes $\mathcal{L}ip_{\mathrm{loc}}, \mathcal{C}, \mathcal{M}$. Then the following are equivalent:*

1. *For every Banach space V, (X,V) supports the p-Poincaré inequality for the class $\mathcal{S}$;*
2. *There exists a Banach space V so that (X,V) supports the p-Poincaré inequality for the class $\mathcal{S}$;*
3. *The space X supports the p-Poincaré inequality for the class $\mathcal{S}$.*

The statement is quantitative in that all the relevant constants depend only on each other, on p, and on the doubling constant of μ.

Remark 4.4. It is clear that the Poincaré inequality for measurable functions implies the corresponding inequality for continuous functions, which in turn implies it for Lipschitz functions. When $V = \mathbb{R}$, the Poincaré inequality for Lipschitz functions implies the corresponding inequality for measurable functions provided the metric space X is *proper* (closed balls in X are compact) and path connected, and the measure μ is doubling. This follows by combining the main result of [25] with the fact that proper and path-connected spaces are quasiconvex if the Poincaré inequality holds [18]. Consequently, by Theorem 4.3, we we obtain a similar assertion for Banach space-valued functions.

In general, the validity of a Poincaré inequality for Lipschitz functions does not imply its validity for continuous functions; see [31]. We do not know what the situation is for measurable versus continuous functions.

Proof of Theorem 4.3. (1) $\Rightarrow$ (2) is trivial. To prove that (2) $\Rightarrow$ (3), fix a ball B in X, a function $f \in \mathcal{S}(\sigma B : \mathbb{R})$, and a p-weak upper gradient ρ of f in σB. Next, fix a vector $e \in V$ with $\|e\| = 1$ and define $F : X \to V$ by $F(x) = f(x)e$. Then $F \in \mathcal{S}(\sigma B : V)$. Moreover, ρ is a p-weak V-upper gradient of F in σB. Finally, $F_B = (f_B)e$, whence the assertion follows.

The proof that (3) $\Rightarrow$ (1) is more involved. We consider (restricted) maximal functions

$$M_{R,p}h(x) := \sup_{0<r<R} \left(\fint_{B(x,r)} |h|^p \, d\mu \right)^{1/p} \tag{4.5}$$

for $0 < R \leq \infty$ and for $h \in L^1_{\mathrm{loc}}(X)$. When $p = 1$ we abbreviate $M_{R,1}h = M_R h$ and when $R = \infty$ we further abbreviate $M_\infty = M$.

The following proposition for $V = \mathbb{R}$ can be found in [18, Theorems 3.2 and 3.3]. The case of a general Banach space is no more difficult and is left to the reader to verify (Proposition 2.10 is needed here).

Proposition 4.6. *Let $X = (X, d, \mu)$ be a doubling metric measure space, and let $\Omega \subset X$ be open. If the pair of functions $F \in L^1_{\mathrm{loc}}(\Omega : V)$ and $\rho : \Omega \to [0, \infty]$ satisfies the p-Poincaré inequality (4.2) for each ball B such that $\sigma B \subset \Omega$, where $\sigma \geq 1$ is fixed, then the inequality*

$$\|F(x) - F(y)\| \leq C d(x,y)(M_{2\sigma d(x,y),p}\rho(x) + M_{2\sigma d(x,y),p}\rho(y)) \tag{4.7}$$

holds for all Lebesgue points x and y of F in Ω, where C depends only on C_P and on the doubling constant C_d.

Conversely, if for some pair of functions F and ρ in Ω as above the pointwise inequality

$$\|F(x) - F(y)\| \leq C d(x,y)(M_{\sigma' d(x,y),p}\rho(x) + M_{\sigma' d(x,y),p}\rho(y)) \tag{4.8}$$

holds, with C and σ' fixed, for almost every $x, y \in \Omega$ such that $x, y \in B$, where B is ball with $3\sigma' B \subset \Omega$, then the pair F, ρ satisfies the p-Poincaré inequality (4.2) for each such B with constants $\sigma = 3\sigma'$ and C_P depending only on p, C, and the doubling constant C_μ.

We now complete the proof of Theorem 4.3 by showing that (3) $\Rightarrow$ (1). Suppose that X satisfies the p-Poincaré inequality for the class $\mathcal{S}$. Fix a Banach space V and a ball B in X. By Proposition 4.6, it suffices to show that (4.8) holds a.e. for all functions $F \in \mathcal{S}(\sigma B : V)$

with p-weak V-upper gradient ρ in the ball σB. We distinguish two cases: (i) $\mathcal{S} = \mathcal{L}ip_{\mathrm{loc}}$ or $\mathcal{S} = \mathcal{C}$ and (ii) $\mathcal{S} = \mathcal{M}$.

(i) Let F be a function either in $\mathcal{L}ip_{\mathrm{loc}}(\sigma B : V)$ or in $\mathcal{C}(\sigma B : V)$ with p-weak V-upper gradient ρ. We shall show that (4.8) holds for each pair of points $x, y \in B$ for some $\sigma' \geq 1$ depending only on the data. To this end, fix $x, y \in B$ (without loss of generality distinct), and choose a functional $v^* \in V^*$ such that $\|v^*\| \leq 1$ and that

$$\langle v^*, F(x) - F(y)\rangle = \|F(x) - F(y)\|.$$

Because $|\langle v^*, F(z)\rangle - \langle v^*, F(w)\rangle| \leq \|F(z) - F(w)\|$ for all $z, w \in \sigma B$, we have that ρ is a p-weak upper gradient of the real-valued function $f(z) = \langle v^*, F(z)\rangle$. Because f is continuous, every point is a Lebesgue point of f, and so (4.8) for the function F follows from the assumptions and from (4.7) applied to the function f.

(ii) Let $F : \sigma B \to V$ be bounded and measurable and let ρ be a p-weak V-upper gradient of F. Choose $Z_0 \subset X$, $\mu(Z_0) = 0$, so that $F(X \setminus Z_0)$ is separable (see Theorem 2.1). Let (v_n), $n \geq 1$, be a countable set that is dense in the difference set $F(X \setminus Z_0) - F(X \setminus Z_0)$. Without loss of generality, we assume that $v_n \neq 0$ for each n. Next, for each n, choose $v_n^* \in V^*$ with $\|v_n^*\| = 1$ and $\langle v_n^*, v_n\rangle = \|v_n\|$, and define $f_n : V \to \mathbb{R}$ by $f_n(z) = \langle v_n^*, F(z)\rangle$. As in (i) above, we have that ρ is a p-weak upper gradient of $f_n \in \mathcal{M}(X : \mathbb{R})$. Therefore, there exists a set $Z_n \subset B$ with $\mu(Z_n) = 0$ so that (4.7) holds (with $F = f_n$) for all $x, y \in B \setminus Z_n$.

Let $Z = \bigcup_{n=0}^{\infty} Z_n$ and let $x, y \in B \setminus Z$. Choose a sequence (v_{n_k}) from our dense set converging to $F(x) - F(y)$. Then, with obvious notation $v_{n_k}^*$,

$$\begin{aligned} \big||f_{n_k}(x) - f_{n_k}(y)| - \|v_{n_k}\|\big| &= |f_{n_k}(x) - f_{n_k}(y) - \langle v_{n_k}^*, v_{n_k}\rangle| \\ &= |\langle v_{n_k}^*, F(x) - F(y) - v_{n_k}\rangle| \leq \|F(x) - F(y) - v_{n_k}\| \to 0 \end{aligned}$$

as $k \to \infty$. Thus $|f_{n_k}(x) - f_{n_k}(y)| \to \|F(x) - F(y)\|$ as $k \to \infty$, and we conclude that (4.8) holds for all $x, y \in B \setminus Z$, where $\mu(Z) = 0$. The proof of Theorem 4.3 is complete. □

5. Equivalence of $N^{1,p}(X : V)$ and the Sobolev space of Korevaar-Schoen

In [30], Korevaar and Schoen define a class of Sobolev maps $F : X \to Y$, where X is a Riemannian domain and Y is a complete metric space. In this section, we show that the Sobolev space $N^{1,p}(X : Y)$ defined in Section 3 and the Korevaar-Schoen space are equivalent in this setting. This result is not original with us; it has been announced by Reshetnyak [45, p. 568]. For the convenience of the reader, we include a proof here. The Sobolev space of Korevaar and Schoen can be defined more generally for abstract metric measure spaces, see the remarks at the end of this section.

We briefly recall the definition of the Korevaar-Schoen Sobolev space [30]. Let Ω be a connected and open subset of a Riemannian n-manifold M such that the metric completion $\overline{\Omega}$ is a compact subset of M; we also assume that Ω has smooth boundary. For $\epsilon > 0$, let

$$\Omega_\epsilon := \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) > \epsilon\}.$$

Let Y be a complete metric space. Let $F : \Omega \to Y$ be a map; we assume that F is measurable (in the sense of Section 2) as a map from Ω to $\ell^\infty(Y)$. For $x, y \in \Omega$, write

$$e_\epsilon(x, y; F) = \frac{d(F(x), F(y))}{\epsilon}.$$

For $1 \le p < \infty$ and $x \in \Omega_\epsilon$ the (nonnormalized) averaged ϵ-approximate density function is

$$e^p_\epsilon(x;F) = \fint_{B(x,\epsilon)} e_\epsilon(x,y;F)^p \, d\mu(y),$$

where $d\mu$ denotes the Riemannian measure on Ω. If $\varphi \in C_c(\Omega,[0,1])$ (i.e., φ is a compactly supported function on Ω taking values in the interval $[0,1]$) and $\epsilon < \operatorname{dist}(\operatorname{supp}(\varphi), \partial\Omega)$, write

$$E^p_\epsilon(\varphi;F) = \int_\Omega \varphi(x) e^p_\epsilon(x;F)\, d\mu(x).$$

Finally, let

$$E^p(F) = \sup_{\varphi \in C_c(\Omega:[0,1])} \limsup_{\epsilon\to 0} E^p_\epsilon(\varphi;F).$$

We say that F is in the *(Korevaar-Schoen) Sobolev space* $KS^{1,p}(\Omega : Y)$ if $E^p(F) < \infty$. By [30, Theorem 1.5.1], if $F \in KS^{1,p}(\Omega : Y)$, then the measures $e^p_\epsilon(\cdot;F)d\mu$ converge weakly to an "energy density" measure $de^p(\cdot;F)$ on Ω with total mass $E^p(F)$.

Theorem 5.1. *Let $\Omega \subset M$ be as above and let Y be a complete metric space. Let $p > 1$. Then $N^{1,p}(\Omega : Y)$ and $KS^{1,p}(\Omega : Y)$ are equal as sets. More precisely, if F is in $N^{1,p}(\Omega : Y)$, then F has a (p-capacity) representative in $KS^{1,p}(\Omega : Y)$ and*

$$E^p(F) \le C(n,p,\Omega) \int_\Omega \rho^p_F \, d\mu. \tag{5.2}$$

Conversely, if $F \in KS^{1,p}(\Omega : Y)$, then F has a Lebesgue representative $\tilde F$ in $N^{1,p}(\Omega : Y)$ and

$$\rho^p_{\tilde F}\, d\mu \le C(n,p) de^p(\cdot;F) \tag{5.3}$$

as measures on Ω. Here $\rho_{\tilde F}$ denotes the minimal p-weak $\ell^\infty(Y)$-upper gradient of $\tilde F$ (as in Section 3).

In the proof of Theorem 5.1 a different version of Theorem 3.17 will be more useful. For simplicity we state only the implication which we need in the proof of Theorem 5.1.

Proposition 5.4. *Let $\Omega \subset M$, Y and $p > 1$ be as in Theorem 5.1 and let $V = \ell^\infty(Y)$. Assume that $F : \Omega \to V$ is a separably valued map for which $\langle v^*, F\rangle \in W^{1,p}(\Omega)$ for all $v^* \in V^*$, $\|v^*\| \le 1$, and assume that there exists $\tilde\rho \in L^p(\Omega)$ so that*

$$|\nabla\langle v^*, F\rangle(x)| \le \tilde\rho(x)$$

for a.e. $x \in \Omega$ and all such v^*. Then F has a Lebesgue representative $\tilde F \in N^{1,p}(\Omega : V)$ and

$$\rho_{\tilde F}(x) \le \sup_n |\nabla\langle v^*_n, F\rangle(x)| \tag{5.5}$$

for a.e. $x \in \Omega$ and some (fixed) countable set of linear functionals (v^*_n) with $\|v^*_n\| \le 1$. Moreover, if $F(x) \in Y$ for μ-a.e. $x \in \Omega$, then $\tilde F(x) \in Y$ for p-quasi-every $x \in \Omega$, i.e., $\tilde F \in N^{1,p}(\Omega : Y)$.

In the interest of brevity, we only sketch the proof of this proposition. First, as in the proof of (3') $\Rightarrow$ (1) in Theorem 3.17, the separability of $F(\Omega)$ yields elements (v_n) and (v^*_n) in V and V^* respectively so that

$$|\langle v^*_n, F\rangle(x) - \langle v^*_n, F\rangle(y)| \le C(n)|x-y|(M\tilde\rho(x) + M\tilde\rho(y))$$

for all $x, y \in \Omega \setminus Z_n$. Here M denotes the (unrestricted) maximal operator and Z_n is a set of zero measure in Ω, cf. Proposition 4.6. By essentially the same argument as in the proof of Theorem 3.17, we see that

$$\|F(x) - F(y)\| \le C(n)|x-y|(M\tilde\rho(x) + M\tilde\rho(y))$$

for a.e. $x, y \in \Omega$. That is, F belongs to the Hajłasz Sobolev space $M^{1,p}(\Omega : V)$ (see the following section for the definition), with

$$\|F\|_{M^{1,p}(\Omega:V)} \le C(n)(\|F\|_{L^p(\Omega:V)} + \|\tilde\rho\|_{L^p(\Omega)}).$$

By the analog of Theorem 6.7 for $M^{1,p}(\Omega : V)$, we may choose a sequence of Lipschitz functions $F_\nu : \Omega \to V$ converging to F in $M^{1,p}(\Omega : V)$. (Observe that $V = \ell^\infty(Y)$ is an absolute Lipschitz retract, whence (Ω, V) has the Lipschitz extension property.) The proof of Lemma 4.7 of [49] guarantees that (F_ν) is a Cauchy sequence in $N^{1,p}(\Omega : V)$ and hence subconverges p-quasiuniformly to a p-quasicontinuous Lebesgue representative $\tilde F$ for F. It remains to show that $\tilde F$ is in $N^{1,p}(\Omega : V)$ which can be seen as follows: since $\tilde F$ is quasicontinuous, so is $\langle v^*, \tilde F\rangle$ for every $v^* \in V^*$. By assumption, $\langle v^*, \tilde F\rangle$ is in $W^{1,p}(\Omega)$, and as a quasicontinuous $W^{1,p}$ function, it also lies in $N^{1,p}(\Omega)$. The fact that $\tilde F \in N^{1,p}(\Omega : V)$ then follows from the implication (3') $\Rightarrow$ (1) of Theorem 3.17 and (5.5) follows from (3.18).

Finally, we verify the last sentence in the statement of Proposition 5.4. Let Z be the set of points $x \in \Omega$ for which $\tilde F(x) \notin Y$. By assumption $F(\Omega)$ is μ-almost contained in Y and hence $\mu(Z) = 0$. Consequently, the collection of all rectifiable curves γ in Ω for which $\tilde F$ is absolutely continuous along γ and $\gamma \cap Z$ has zero length is a collection of full p-modulus. A straightforward argument involving the absolute continuity of $\tilde F$ along γ and the completeness of Y shows that every such curve in fact lies in $\Omega \setminus Z$. In other words, the collection of all rectifiable curves in Ω which meet Z has p-modulus zero, equivalently, Z has zero p-capacity in Ω. This completes the proof that $\tilde F \in N^{1,p}(\Omega : Y)$.

Proof of Theorem 5.1. Assume first that $F \in N^{1,p}(\Omega : Y)$. Then $F \in N^{1,p}(\Omega : \ell^\infty(Y))$ and $F(x) \in Y$ for p-quasi-every $x \in X$; upon changing F on a set of p-capacity zero, we may assume that $F(X) \subset Y$. Let $\rho_F \in L^p(\Omega)$ denote the minimal p-weak $\ell^\infty(Y)$-upper gradient of F in Ω.

Since $\Omega \subset M$ is precompact and $\partial\Omega$ is smooth, the 1-Poincaré inequality holds in Ω. By Proposition 4.6,

$$d(F(x), F(y)) \le C(n, \Omega)|x-y|(M\rho_F(x) + M\rho_F(y))$$

for a.e. $x, y \in \Omega$, where Mh denotes the (unrestricted) Hardy-Littlewood maximal function of $h \in L^1_{\mathrm{loc}}(\Omega)$. It follows that for $\epsilon > 0$ and $x \in \Omega_\epsilon$,

$$\begin{aligned} e^p_\epsilon(x; F) &= \fint_{B(x,\epsilon)} \frac{d(F(x), F(y))^p}{\epsilon^p} d\mu(y) \\ &\le C(n, \Omega)\, M\rho_F(x)^p + C(n, \Omega) \fint_{B(x,\epsilon)} M\rho_F(y)^p \, d\mu(y). \end{aligned}$$

For $\varphi \in C_c(\Omega : [0,1])$ and $\epsilon < \frac{1}{2}\operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega)$,

$$\begin{aligned} E_\epsilon^p(\varphi; F) &= \int_\Omega \varphi(x) e_\epsilon^p(x; F)\, d\mu(x) \\ &\le C(n,\Omega) \int_\Omega M\rho_F(x)^p\, d\mu(x) + C(n,\Omega) \int_{\Omega_{2\epsilon}} \fint_{B(x,\epsilon)} M\rho_F(y)^p\, d\mu(y) d\mu(x) \\ &\le C(n,\Omega) \int_\Omega (M\rho_F)^p\, d\mu, \end{aligned}$$

where the last inequality follows by applying Fubini's theorem to the second term and using the doubling property of μ. Since $p > 1$, we may apply the Hardy-Littlewood Maximal Theorem to conclude that

$$E_\epsilon^p(\varphi; F) \le C(n,p,\Omega) \int_\Omega \rho_F^p\, d\mu.$$

Letting ϵ tend to zero and taking the supremum over all φ shows that $F \in KS^{1,p}(\Omega : Y)$ and that (5.2) holds.

Now assume that $F \in KS^{1,p}(\Omega : Y) \subset KS^{1,p}(\Omega : V)$, where $V = \ell^\infty(Y)$. By Theorem 2.1, F is essentially separably valued; upon changing F on a set of measure zero we may assume that F is separably valued. Next, the energy density $de^p(\cdot; F)$ is a Radon measure on Ω (since it arises as the weak limit of the measures $e_\epsilon^p(\cdot; F)d\mu$ and each of the functions $e_\epsilon^p(\cdot; F)$, $\epsilon > 0$, is measurable). For $x \in \Omega$, let

$$h(x) = \lim_{r\to 0} \frac{e^p(B(x,r); F)}{\mu(B(x,r))} \tag{5.6}$$

denote the Lebesgue derivative of the measure $e^p(\cdot; F)$ with respect to the Riemannian measure μ (see, for example [38, Definition 2.9]). By the Lebesgue differentiation theorem [38, Theorem 2.12], $h(x)$ exists and is finite for μ-a.e. point x in Ω; moreover,

$$h\, d\mu \le de^p(\cdot; F) \tag{5.7}$$

as measures on Ω.

By [30, Corollary 1.6.3], $\langle v^*, F\rangle$ belongs to the standard Sobolev space $W^{1,p}(\Omega)$ for each $v^* \in V^*$, $\|v^*\| \le 1$, and

$$|\nabla\langle v^*, F\rangle|^p\, d\mu \le C(n,p) de^p(\cdot; F) \tag{5.8}$$

as measures. Differentiating with respect to μ in (5.8) and using the Lebesgue Differentiation Theorem on the left hand side and (5.6) on the right hand side, we see that

$$|\nabla\langle v^*, F\rangle(x)|^p \le C(n,p) h(x) \tag{5.9}$$

for μ-a.e. x in Ω.

Since $e^p(\cdot, F)$ has finite total mass on Ω, (5.7) implies that $h \in L^1(\Omega, \mu)$, whence condition (2') of Corollary 3.21 is satisfied. By Proposition 5.4, F has a Lebesgue representative $\tilde F \in N^{1,p}(\Omega : V)$. (Note that $\tilde F \in L^p(\Omega : V)$ because $\partial\Omega$ is smooth, by standard embedding theorems.) Moreover, by the last part of Proposition 5.4, $\tilde F \in N^{1,p}(\Omega : Y)$. Finally, by (5.5),

$$\rho_{\tilde F}(x) \le \sup_n |\nabla\langle v^*, F\rangle(x)| \tag{5.10}$$

for μ-a.e. $x \in \Omega$, where the supremum is taken over a countable set of linear functionals $v^* \in V^*$, $\|v^*\| \le 1$. Now (5.3) follows by combining (5.7), (5.9) and (5.10). □

Remarks 5.11. More generally, we may consider the Korevaar-Schoen definition in the case when the source domain is replaced by an abstract metric measure space. In this setting, it is more convenient to work with a slight modification of the definition of $KS^{1,p}$. For a map $F : X \to Y$ from a metric measure space $X = (X, d, \mu)$ into a complete metric space Y, we define $e^p_\epsilon(x; F)$, $x \in X$, $\epsilon > 0$, $p \geq 1$, as before and set

$$E^p(F) = \sup_B \left(\limsup_{\epsilon \to 0} \int_B e^p_\epsilon(x; F)\, d\mu(x) \right),$$

where the supremum is taken over all metric balls in X. Then F is said to be in the Korevaar-Schoen Sobolev space $KS^{1,p}(X : Y)$ if $E^p(F)$ is finite. Clearly this definition agrees with that of Korevaar and Schoen in the case when X is proper, i.e., closed balls in X are compact.

If the metric measure space (X, μ) is doubling (Remark 2.8) and satisfies the p-Poincaré inequality for measurable functions (Definition 4.1) with $p > 1$, then

$$N^{1,p}(X : V) = H^{1,p}(X : V) \subset KS^{1,p}(X : V) = P^{1,p}(X : V); \tag{5.12}$$

we assume here for simplicity that the target is a Banach space. The space $H^{1,p}(X : V)$ in (5.12) denotes Cheeger's Sobolev space (see Remark 3.8), consisting of those functions $F \in L^p(X : V)$ for which there exists a sequence $F_i \in L^p(X : V)$ with corresponding upper gradients $\rho_i \in L^p(X)$ so that (F_i) converges to F in $L^p(X : V)$ and $\sup_i \|\rho_i\|_p < \infty$. Also $P^{1,p}(X : V)$ denotes the set of functions $F : X \to V$ for which there exists a real-valued function $\rho \in L^p(X)$ so that the pair of functions F and ρ satisfies the p-Poincaré inequality (4.2) in each ball B of X, with the constant C_P independent of B.

The inclusions in (5.12) can be proved as in [32] and [49]. More precisely, the inclusion $N^{1,p}(X : V) \subset P^{1,p}(X : V)$ is trivial, the inclusion $P^{1,p}(X : V) \subset KS^{1,p}(X : V)$ follows by modifying the argument of Theorem 4.1 of [32], and the inclusion $KS^{1,p}(X : V) \subset P^{1,p}(X : V)$ is (essentially) contained in Theorem 4.5 of [32]. The equivalence of $N^{1,p}(X : V)$ and $H^{1,p}(X : V)$ has already been mentioned for $V = \mathbb{R}$ in Remark 3.8; the general case is similar. Note that in Theorem 4.5 of [32] it is assumed that X satisfies the q-Poincaré inequality for some $q < p$; this assumption, however, is only used to show the equivalence with yet another notion of Sobolev space due to Hajłasz, which we discuss in the following section. See Remarks 6.12 for a discussion about equality in (5.12).

6. Embedding theorems. Density of Lipschitz maps and quasicontinuity.

In this section, we briefly indicate what classical embedding theorems hold true for the Sobolev space $N^{1,p}(X : V)$ as defined in 3.9. We also discuss the density of Lipschitz functions in $N^{1,p}(X : V)$ and the quasicontinuity of $N^{1,p}(X : V)$ functions. In light of the theory presented in Sections 3 and 4, the discussion here reduces to previous works in the area.

Throughout this section, $X = (X, d, \mu)$ is a metric measure space as in 1.2. Recall also the definition of the Poincaré inequality from 4.1.

We say that the measure μ satisfies a *local lower mass bound with exponent* $Q > 0$ if there exist constants $C_0 \geq 1$ and $R_0 > 0$ so that

$$\frac{\mu(B_R)}{\mu(B_{R_0})} \geq C_0^{-1} (\frac{R}{R_0})^Q \tag{6.1}$$

whenever $B_R \subset B_{R_0}$ are balls in X with $\text{radius}(B_R) = R \leq R_0 = \text{radius}(B_{R_0})$. It is easy to see that every doubling measure satisfies (6.1) for some $Q > 0$.

Theorem 6.2. *Assume that μ is a doubling measure on X satisfying the local lower mass bound (6.1) for some exponent $Q > 1$ and assume that X supports the p-Poincaré inequality. Let V be a Banach space and let $F \in N^{1,p}_{\mathrm{loc}}(X : V)$ be such that $F \in N^{1,p}(B : V)$ for all balls $B \subset X$ of radius $R < R_0$. If $1 \le p < Q$ and $1 \le q < p^* = Qp/(Q-p)$, then*

$$(6.3)\qquad \Big(\fint_B \|F - F_B\|^q \, d\mu\Big)^{1/q} \le C(\operatorname{diam} B)\Big(\fint_{5\sigma B} \rho^p \, d\mu\Big)^{1/p}$$

for all balls B in X of radius $R < R_0$ and all V-upper gradients ρ of F in X. If $p > Q$, then F has a representative that is locally Hölder continuous in the following sense:

$$(6.4)\qquad \|F(x) - F(y)\| \le C(\operatorname{diam} B)^{Q/p} d(x,y)^{1-Q/p}\Big(\fint_{5\sigma B} \rho^p \, d\mu\Big)^{1/p}$$

for all balls B in X of radius $R < R_0$, all pairs of points $x, y \in B$, and all V-upper gradients ρ of F in X. Here $C \ge 1$ in both (6.3) and (6.4) is a constant depending only on the data of the setting, and $\sigma \ge 1$ as in (4.2).

The discussion in the previous sections understood, Theorem 6.2 follows from the proof of [18, Theorem 5.1]. More precisely, let $F \in N^{1,p}_{\mathrm{loc}}(X : V)$ be as in the theorem, let ρ be a V-upper gradient of F in X, and let $B \subset X$ be a ball. Because a.e. point of X is a Lebesgue point of F (Proposition 2.10), we obtain by use of the p-Poincaré inequality as in [18, Theorem 5.2] that the inequality

$$(6.5)\qquad \|F(x) - F_B\| \le C J^{\sigma,B}_{1,p}\rho(x)$$

holds for a.e. x in B, where

$$J^{\sigma,B}_{1,p}\rho(x) = \sum_{\substack{i \\ 2^i \le 2\sigma \operatorname{diam}(B)}} 2^i \Big(\fint_{B(x,2^i)} \rho^p \, d\mu\Big)^{1/p}$$

is a generalized Riesz potential. The claimed inequalities (6.3) and (6.4) now follow from the mapping properties of the potential $J^{\sigma,B}_{1,p}$ proved in [18, Theorem 5.3].

Remark 6.6. The crucial estimate here is (6.5). From it one obtains more precise embedding information along the lines of [18, Section 5]. We have, for example, that (locally) $N^{1,p} \hookrightarrow \text{weak} - L^{p^*}$, and that a Trudinger-type inequality is valid in the borderline ($p = Q$) case.

We do not know, however, if the (local) embedding $N^{1,p} \hookrightarrow L^{p^*}$ holds under the assumptions of Theorem 6.2. It holds for real-valued Sobolev mappings; see [18, 5.1(22)].

Next, we discuss the density of Lipschitz functions in the class $N^{1,p}(X : V)$ and the quasicontinuity of elements of $N^{1,p}(X : V)$.

We say that a pair of metric spaces (X, Y) has the *Lipschitz extension property* if there exists a constant $C \ge 1$ so that whenever $E \subset X$ and $f : E \to Y$ is L-Lipschitz, then f has a CL-Lipschitz extension $\tilde{f} : X \to Y$. This extension problem has been studied extensively. It is easy to see, by using the classical McShane extension, that the pair $(X, \ell^\infty(A))$ has the Lipschitz extension property (with $C = 1$) for any space A. If the space Y is an absolute Lipschitz retract, then (X, Y) has the Lipschitz extension property for every metric space X. See [5].

Theorem 6.7. *Let $X = (X, \mu)$ be a doubling metric measure space supporting the p-Poincaré inequality, $1 \leq p < \infty$, and let V be a Banach space for which the pair (X, V) has the Lipschitz extension property. Then Lipschitz maps from X to V are dense in the class $N^{1,p}(X : V)$.*

The proof of this theorem follows along the (now standard) lines of [49, Theorem 4.1] or [9, Theorem 4.24]. We leave the details to the reader.

Corollary 6.8. *Let X be as in the previous theorem and let Y be an arbitrary metric space. Then each function in $N^{1,p}(X : Y)$ is p-quasicontinuous, i.e., continuous on the complement of subsets of X of arbitrarily small p-capacity.*

Recall that we defined the Sobolev space $N^{1,p}(X : Y)$ for maps between two metric spaces to be the collection of maps F in $N^{1,p}(X : \ell^\infty(Y))$ for which $F(x) \in Y$ for p-quasi-every $x \in X$. Thus in the setting of Corollary 6.8 every map from $N^{1,p}(X : Y)$ can be approximated by Lipschitz maps in $N^{1,p}(X : \ell^\infty(Y))$. Corollary 6.8 follows by choosing a sequence of Lipschitz maps which converge back to the original function uniformly in the complement of sets of arbitrarily small p-capacity.

Remark 6.9. It is an interesting problem to determine when one can choose the Lipschitz approximants to have values in the target Y. The proof in [49] shows that this can be done provided the pair (X, Y) has the Lipschitz extension property; nontrivial examples of such pairs have recently been exhibited by Lang, Pavlović, and Schroeder [35]. The case of maps between two compact Riemannian manifolds was settled by Bethuel [6]; the answer depends on the algebraic topology of the image. In the general case many interesting questions remain open. For instance, one may ask to what extent Bethuel's results have analogs for general spaces.

In [17], Hajłasz defined Sobolev spaces $M^{1,p}(X)$, $1 \leq p < \infty$, for an arbitrary metric measure space X by requiring that a function $f \in M^{1,p}(X)$ if and only if $f \in L^p(X)$ and there exists a nonnegative function $g \in L^p(X)$ so that

$$|f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \tag{6.10}$$

for a.e. $x, y \in X$. We endow $M^{1,p}(X)$ with the norm $\|f\|_{M^{1,p}} := \|f\|_p + \inf_g \|g\|_p$, where the infimum is taken over all $g \in L^p(X)$ for which (6.10) is satisfied a.e. Hajłasz showed that the space $M^{1,p}(\Omega)$ is isomorphic to the standard Sobolev space $W^{1,p}(\Omega)$ whenever $1 < p < \infty$ and Ω is a smooth bounded domain in Euclidean space. Hajłasz's definition (6.10) extends in a straightforward manner to maps with target an arbitrary metric space Y; let us denote the resulting class of maps by $M^{1,p}(X : Y)$. Troyanov [50] has studied the relationship between Hajłasz's approach and Reshetnyak's approach to Sobolev maps between metric spaces. See also [3].

The following is an easy consequence of [49, Section 4] and the results of this section.

Theorem 6.11. *Let $X = (X, \mu)$ be a doubling metric measure space of finite mass supporting the q-Poincaré inequality, $1 \leq q < \infty$, and let V be a Banach space for which the pair (X, V) has the Lipschitz extension property. Then for each $p > q$, the space $M^{1,p}(X : V)$ is isomorphic (as a Banach space) to $N^{1,p}(X : V)$. In particular, if Y is a metric space, then $M^{1,p}(X : Y)$ and $N^{1,p}(X : Y)$ are equal (as sets).*

Remarks 6.12. We continue to assume that X, V and p are as in the assumptions of Theorem 6.11, i.e., that X supports a slightly better Poincaré inequality ($q < p$). By following the proof of Theorem 4.5 of [32], we see that in this case the Korevaar-Schoen space $KS^{1,p}(X:V)$ embeds in the Hajłasz space $M^{1,p}(X:V)$. By combining this with the observations in Remark 5.11, we see that all five spaces $N^{1,p}$, $H^{1,p}$, $KS^{1,p}$, $P^{1,p}$, and $M^{1,p}$ coincide in this case. Moreover, there exists $C \geq 1$ depending only on the data of X and on p so that

$$\frac{1}{C}\int_X \rho_F(x)^p \, d\mu(x) \leq E^p(F) \leq C\int_X \rho_F(x)^p \, d\mu(x)$$

whenever $F : X \to Y$.

If $V = \mathbb{R}$, the equality of the first four spaces in this list follows from the reflexivity of $N^{1,p}(X) = H^{1,p}(X)$ and Mazur's lemma. We do not know whether this argument works in the general Banach space-valued setting, or whether $N^{1,p}(X:V) = KS^{1,p}(X:V)$ (for general metric measure spaces X and Banach spaces V) if we only assume that X satisfies the p-Poincaré inequality.

7. Absolute continuity of Sobolev mappings

In [36], Malý and Martio called a map $F : \Omega \to \mathbb{R}^n$, where Ω is a domain in $\mathbb{R}^n$, *K-pseudomonotone*, $K \geq 1$, if $\operatorname{diam} F(B(x,r)) \leq K \operatorname{diam} F(\partial B(x,r))$ for all $x \in \Omega$ and all $r < \operatorname{dist}(x, \partial\Omega)$. They showed that each continuous pseudomonotone mapping F in the Sobolev class $W^{1,n}(\Omega)$ satisfies Lusin's condition N, that is, F is absolutely continuous in measure. Because homeomorphisms between Euclidean domains are monotone, and because quasiconformal homeomorphisms between domains in $\mathbb{R}^n$ are locally in $W^{1,n}$, one obtains in particular the absolute continuity in measure of quasiconformal maps in Euclidean domains. In this section, we prove an extension of the Malý-Martio theorem for Banach space-valued pseudomonotone Sobolev maps, defined on a doubling metric measure space with Poincaré inequality.

Assume that (X, μ) is a metric measure space and that V is a Banach space. Following Malý and Martio, we call a map $F : X \to V$ *pseudomonotone* if there is $K \geq 1$ and $r_0 > 0$ so that

$$\operatorname{diam} F(B(x,r)) \leq K \operatorname{diam} F(\partial B(x,r)) \tag{7.1}$$

for all $x \in X$ and all $0 < r < r_0$.

Theorem 7.2. *Assume that μ is a doubling measure on X satisfying the local lower mass bound (6.1) for some exponent $Q > 1$. Assume further that X supports the Q-Poincaré inequality for continuous functions. Then every continuous pseudomonotone Sobolev map $F \in N^{1,Q}_{\mathrm{loc}}(X:V)$ satisfies Lusin's condition N in the following sense: if $E \subset X$ and $\mu(E) = 0$, then $\mathcal{H}_Q(F(E)) = 0$.*

Here (and below) $\mathcal{H}_Q$ denotes the Hausdorff Q-measure in V.

Note that we do not stipulate any condition for the Hausdorff dimension of X (although (6.1) implies that it is at most Q). Also note that the issue is completely local, so it suffices to assume that something like (6.1) holds locally (that is, not necessarily uniformly for balls of radius less than some fixed R_0).

Proof. Let $E \subset X$ satisfy $\mu(E) = 0$. Without loss of generality, we may assume that E belongs to a fixed ball of radius $R_1 < \min\{r_0, R_0\}$, where r_0 and R_0 are given in (7.1) and

(6.1), respectively. Moreover, we may assume that F has an upper gradient ρ that is Q-integrable in some neighborhood of E. Fix $\epsilon > 0$. Then fix an open superset $\Omega \supset E$ in X such that

$$\int_{\Omega} \rho^Q \, d\mu < \epsilon\,. \tag{7.3}$$

Next, fix a number $\Lambda > 1$, to be determined later. (Λ will only depend on the data of the problem, and in particular not on ϵ.) We define two sets H_Λ and P_Λ as follows: a point x is in H_Λ if and only if $x \in E$ and there are arbitrarily small radii $r > 0$ such that

$$\int_{B(x,10\sigma r)} \rho^Q \, d\mu < 2\Lambda \int_{B(x,r/5)} \rho^Q \, d\mu\,, \tag{7.4}$$

where $\sigma \geq 1$ is the constant appearing in the Poincaré inequality (4.2). Put $P_\Lambda = E \setminus H_\Lambda$.

Let us first show that $\mathcal{H}_Q(F(H_\Lambda)) = 0$. Indeed, for each $x \in H_\Lambda$ we can find $r_x > 0$ such that $B(x, 10\sigma r_x) \subset \Omega$ and that (7.4) holds for $r = r_x$. It follows from an abstract Sobolev embedding theorem on spheres, [18, Theorem 7.1], that there exists a radius $r \in (r_x, 2r_x)$ so that

$$\|F(z) - F(w)\| \leq C d(z,w)^{1/Q} r_x^{1-1/Q} \Big(\fint_{B(x,10\sigma r_x)} \rho^Q \, d\mu\Big)^{1/Q} \tag{7.5}$$

for each $z, w \in \Omega$ with $d(z,x) = r = d(w,x)$. Here, and in what follows, $C \geq 1$ denotes a positive constant that depends only on the data of the problem, and in particular is independent of x and r_x.

In fact, [18, Theorem 7.1] is stated and proved for real-valued functions only, but it is routine to check that the argument continues to work when the target is a Banach space. (This result makes use of the Lebesgue differentiation theorem 2.10 for Banach space-valued maps.)

Next, because F is pseudomonotone, and because μ satisfies (6.1), we obtain from (7.5) that

$$(\operatorname{diam} F(B(x,r_x)))^Q \leq C(\operatorname{diam} F(\partial B(x,r)))^Q \leq C \int_{B(x,10\sigma r_x)} \rho^Q \, d\mu\,. \tag{7.6}$$

Thus $H_\Lambda \subset \cup_{x \in H_\Lambda} B(x, r_x)$, where $B(x, r_x)$ satisfies (7.6). By standard covering theorems, we can choose a countable collection $\{B(x_i, r_i)\}$ of balls such that their union covers H_Λ, that the collection $\{B(x_i, r_i/5)\}$ is pairwise disjoint, and that both

$$(\operatorname{diam} F(B(x_i,r_i)))^Q \leq C \int_{B(x_i,10\sigma r_i)} \rho^Q \, d\mu$$

and (7.4) (for $r = r_i$) holds for each ball from the collection. We deduce that

$$\begin{aligned} \sum_i (\operatorname{diam} F(B(x_i,r_i)))^Q &\leq C \sum_i \int_{B(x_i,10\sigma r_i)} \rho^Q \, d\mu \\ &\leq 2\Lambda C \sum_i \int_{B(x_i,r_i/5)} \rho^Q \, d\mu \leq 2\Lambda C \int_\Omega \rho^Q \, d\mu \leq 2\Lambda C \epsilon\,. \end{aligned}$$

Because $F(H_\Lambda) \subset \cup_i F(B(x_i, r_i))$, we have by letting $\epsilon \to 0$ that $\mathcal{H}_Q(F(H_\Lambda)) = 0$.

Next we prove that $\mathcal{H}_Q(F(P_\Lambda)) = 0$. To this end, it suffices to show that $\mathcal{H}_Q(F(P^m)) = 0$, where $P^m = P^m_\Lambda$ consists of those $x \in P_\Lambda$ for which

$$2\Lambda \int_{B(x,r/5)} \rho^Q \, d\mu \le \int_{B(x,10\sigma r)} \rho^Q \, d\mu \,, \qquad 0 < r < 1/m \,. \tag{7.7}$$

Moreover, it suffices to show that

$$\mathcal{H}_Q(F(P^m \cap B(x_0, 1/10m))) = 0 \tag{7.8}$$

for a given arbitrary $x_0 \in P^m$. Fix $x, y \in P^m \cap B(x_0, 1/10m)$. Then $2d(x,y) < 1/m$ and hence

$$\|F(x) - F(y)\| \le C d(x,y)(M_{1/m,Q}\rho(x) + M_{1/m,Q}\rho(y)) \tag{7.9}$$

by Lemma 4.6. (Recall the definition for the restricted maximal function $M_{1/m,Q}$ from (4.5), and observe that every point is a Lebesgue point of F because F is continuous.) To estimate $M_{1/m,Q}\rho(x)$, fix $0 < r < 1/m$. Let $k \ge 0$ be the integer for which $\sigma_0^{-k-1} \le mr < \sigma_0^{-k}$, where $\sigma_0 = 50\sigma$. We obtain from (7.7) and from the doubling property of μ that

$$\begin{aligned} \fint_{B(x,r)} \rho^Q \, d\mu &\le C \fint_{B(x,\sigma_0^{-k}/m)} \rho^Q \, d\mu \\ &\le C(\mu,\sigma)^k \Lambda^{-k} \fint_{B(x,1/m)} \rho^Q \, d\mu \le C(\mu,\sigma,m) < \infty \end{aligned}$$

if $\Lambda > 1$ is chosen large enough (depending only on μ and σ). It follows that $M_{1/m,Q}\rho(x) \le C(\mu,\sigma,m)$, and similarly for $M_{1/m,Q}\rho(y)$. Because x and y were arbitrary points in $P^m \cap B(x_0, 1/10m)$, we deduce from (7.9) that F is Lipschitz on $P^m \cap B(x_0, 1/10m)$. Finally, because $\mu(E) = 0$ implies $\mathcal{H}_Q(E) = 0$ by condition (6.1), we find that (7.8) holds.

The proof of Theorem 7.2 is complete. □

Remark 7.10. The above proof shows that under the assumptions of Theorem 7.2, the Hausdorff Q-measure of $F(E) \subset V$ is locally finite if the Hausdorff Q-measure of $E \subset X$ is locally finite, where $Q > 1$ is as in (6.1). In particular, if X has locally finite Hausdorff Q-measure, then $F(X)$ does, too. As illustrated by (now) standard examples in $\mathbb{R}^n$, the assumption of pseudomonotonicity of F is necessary in Theorem 7.2. See [36, Section 5].

8. Quasisymmetric mappings and Sobolev spaces

An embedding $F : X \to V$ of a metric space X in a Banach space V is called *quasisymmetric* if there exists an increasing homeomorphism $\eta : [0,\infty) \to [0,\infty)$ so that

$$\frac{\|F(x) - F(y)\|}{\|F(x) - F(z)\|} \le \eta\left(\frac{d(x,y)}{d(x,z)}\right)$$

for all $x, y, z \in X$, $x \ne z$. Naturally, F can be an embedding of X in some other metric space Y, but as pointed out several times in this paper, Y can always be thought of as a subset of a Banach space. See [51] for the basic theory of quasisymmetric maps between metric spaces.

Our goal in this section is to show that F is in the local Sobolev space $N^{1,Q}_{\mathrm{loc}}(X : V)$, provided that X is locally compact, Ahlfors Q-regular, and supports the Q-Poincaré inequality, and provided that $F(X)$ has locally finite Hausdorff Q-measure (Theorem 8.12). A key point to our argument is an approximation of the map F via a sequence of "discrete convolution approximations". We note that this approximation procedure would not work unless it is assumed that the target space has a linear structure of some sort.

We shall then use this fact to give two different proofs of the absolute continuity in measure of F. The first proof is simply a reduction to Theorem 7.2, by showing that quasisymmetric maps are pseudomonotone in our setting. The second proof uses the modulus of curve families and is related to the proof given in pp. 111-112 of [56].

Let $F : X \to V$ be a quasisymmetric embedding of a metric space X in a Banach space V. Assume that X has the following *doubling property*: there is a constant $C \geq 1$ so that no ball B in X can contain more than C disjoint balls of half the radius of B. For example, if X carries a doubling measure, then X has the doubling property; the converse need not be true. We begin by constructing discrete convolution approximations F_ϵ to F, following the discussion on pp. 290-292 of [46].

Fix $\epsilon > 0$. Choose an ϵ-net in X, that is, a countable collection of points (x_i) in X so that $d(x_i, x_j) \geq \epsilon$ whenever $i \neq j$ and $X = \cup_i B_i$, where $B_i = B(x_i, \epsilon)$ is the open ball centered at x_i with radius ϵ. Because X has the doubling property, the dilated balls $2B_1, 2B_2, \ldots$ have *bounded overlap*: the sum of the characteristic functions of the balls $2B_i$ is uniformly bounded from above by a constant depending only on the doubling constant of X (and not on the value of ϵ). Moreover, the family $\{\frac{1}{2}B_i\}$ is pairwise disjoint.

There exists a *locally finite Lipschitz partition of unity*, subordinate to the cover $\{2B_i\}$ of X. That is, for each i, we may choose a function $\varphi_i : X \to [0,1]$ with the following properties:

(i) $\sum_i \varphi_i \equiv 1$;
(ii) $\varphi_i \equiv 0$ on $X \setminus 2B_i$;
(iii) φ_i is C/ϵ-Lipschitz, where C depends only on the doubling constant of X.

The family of functions $\{\varphi_i\}$ is locally finite (at most finitely many of these functions are nonzero at each point of X) and so only finitely many terms in the sum in (i) are nonzero.

We define the *discrete convolution approximation* $F_\epsilon : X \to V$ *of* F *at the level* ϵ by

$$F_\epsilon(x) = \sum_i \varphi_i(x) F(x_i).$$

In the following, we understand that an ϵ-net $(x_i) = (x_i^\epsilon)$ together with the associated balls $B_i^\epsilon = B(x_i, \epsilon)$ have been given for each $\epsilon > 0$.

Lemma 8.1. *The function F_ϵ is Lipschitz on bounded sets and hence is in $N^{1,Q}_{\mathrm{loc}}(X : V)$ for sufficiently small $\epsilon > 0$.*

Proof. Let $U \subset X$ be bounded. Since X has the doubling property, only finitely many of the balls $2B_i$ can meet U; let M_U be the sum of the values $\|F(x_i)\|$ over all of these indices i. If $x, y \in U$, then

$$\|F_\epsilon(x) - F_\epsilon(y)\| \leq \sum_{\substack{i \\ \{x,y\} \cap 2B_i \neq \emptyset}} |\varphi_i(x) - \varphi_i(y)| \cdot \|F(x_i)\| \leq \frac{C}{\epsilon} M_U \, d(x,y),$$

and the lemma follows by Example 3.15. $\square$

Lemma 8.2. *The functions F_ϵ converge to F uniformly on bounded sets.*

Proof. Again, let U be bounded in X. Since F is quasisymmetric, it is uniformly continuous on the set

$$\tilde{U} := \{x \in X : \operatorname{dist}(x, U) < 5\epsilon\};$$

let $\omega_{\tilde{U}}$ be a modulus of continuity. Next, let $x \in U$, and choose a ball B_j from the collection $\{B_i\}$ such that B_j contains x. Then

$$\|F_\epsilon(x) - F(x)\| \le \sum_{\substack{i \\ 2B_i \cap 2B_j \neq \emptyset}} \|F(x_i) - F(x)\| \cdot \varphi_i(x) \le \omega_{\tilde{U}}(5\epsilon)$$

and the lemma follows. □

Next, define a function $\rho_\epsilon : X \to \mathbb{R}$ by

$$\rho_\epsilon(x) = \sum_i \frac{\operatorname{diam} F(B_i^\epsilon)}{\epsilon} \chi_{B_i^\epsilon}(x),$$

where $B_i^\epsilon = B(x_i^\epsilon, \epsilon)$. Clearly, ρ_ϵ is a Borel function.

Lemma 8.3. *Assume that X is connected. There exists a constant $C \ge 1$, independent of ϵ, so that $C \cdot \rho_\epsilon$ is a V-upper gradient of F_ϵ.*

Proof. Fix $\epsilon > 0$. For simplicity, write $B_i^\epsilon = B_i$ and $x_i^\epsilon = x_i$, etc. By remarks in Example 3.15, it suffices to show that $\operatorname{Lip} F_\epsilon \le C \cdot \rho_\epsilon$. Let $x \in X$ and choose a ball B_j from the collection $\{B_i\}$ such that B_j contains x. Quasisymmetry of F, together with the connectedness of X, implies that there exists a constant $C \ge 1$, independent of ϵ, so that

$$\rho_\epsilon(x) \ge \frac{1}{C} \frac{\operatorname{diam} F(5B_j)}{\epsilon}.$$

Now

$$\begin{aligned}
\operatorname{Lip} F_\epsilon(x) &= \limsup_{r \to 0} r^{-1} \sup_{\substack{y \\ d(x,y) \le r}} \|F_\epsilon(x) - F_\epsilon(y)\| \\
&= \limsup_{r \to 0} r^{-1} \sup_{\substack{y \\ d(x,y) \le r}} \|\sum_i (\varphi_i(x) - \varphi_i(y)) \left(F(x_i) - F(x_j)\right)\| \\
&\le \limsup_{r \to 0} r^{-1} \sup_{\substack{y \in B_j \\ d(x,y) \le r}} \sum_{\substack{i \\ 2B_i \cap 2B_j \neq \emptyset}} |\varphi_i(x) - \varphi_i(y)| \cdot \|F(x_i) - F(x_j)\| \\
&\le \frac{C}{\epsilon} \limsup_{r \to 0} r^{-1} \sup_{\substack{y \in B_j \\ d(x,y) \le r}} d(x,y) \sum_{\substack{i \\ 2B_i \cap 2B_j \neq \emptyset}} \|F(x_i) - F(x_j)\| \\
&\le \frac{C}{\epsilon} \operatorname{diam} F(B(x_j, 5\epsilon)) \le C \rho_\epsilon(x).
\end{aligned}$$

The lemma follows. □

The following is a key proposition:

Proposition 8.4. *Let (X, μ) be a connected doubling metric measure space and assume that there are constants $C \ge 1$ and $Q \ge 1$ so that*

$$\mu(B_R) \le C R^Q \tag{8.5}$$

whenever B_R is a ball in X of radius $0 < R < \operatorname{diam} X$. Assume that $F : X \to V$ is a quasisymmetric embedding of X in a Banach space V for which the Hausdorff Q-measure on $F(X)$ is both locally finite and satisfies

$$\mathcal{H}_Q(B_R \cap F(X)) \geq C^{-1} R^Q \tag{8.6}$$

for each ball B_R in V of radius $0 < R < \operatorname{diam} F(X)$ and for some $C \geq 1$ independent of the ball. Then the functions ρ_ϵ are in $L^Q_{\mathrm{loc}}(X)$ uniformly in the following sense: each point in X has a neighborhood U so that

$$\sup_{0<\epsilon\leq\epsilon_0(U)} \int_U \rho_\epsilon^Q \, d\mu < \infty.$$

Recall that local finiteness of the Hausdorff Q-measure means that every point in X has a neighborhood U with $\mathcal{H}_Q(F(U)) < \infty$.

Proof. Let $U \subset X$ be a bounded set for which the Hausdorff Q-measure of $F(\tilde{U})$ is finite, where $\tilde{U} := \{x \in X : \operatorname{dist}(x, U) < \operatorname{diam} U\}$. Fix $\epsilon \leq \frac{1}{10} \operatorname{diam} U$. We may cover U with finitely many of the balls $B_i = B_i(x_i, \epsilon)$ from the covering associated with ϵ. Denote these balls $\hat{B}_1, \ldots, \hat{B}_N$; we may clearly assume that they all meet U. We have

$$\begin{aligned}\int_{\hat{B}_j} \rho_\epsilon^Q \, d\mu &\leq C \sum_{\substack{i \\ B_i \cap \hat{B}_j \neq \emptyset}} \epsilon^{-Q} \int_{\hat{B}_j} (\operatorname{diam} F(B_i))^Q \, d\mu \\ &\leq C(\operatorname{diam} F(5\hat{B}_j))^Q \leq C(\operatorname{diam} F(\hat{B}_j))^Q\end{aligned} \tag{8.7}$$

by (8.5) and by quasisymmetry of F. Further, because the balls $\frac{1}{2}\hat{B}_1, \ldots, \frac{1}{2}\hat{B}_N$ are disjoint, quasisymmetry of F implies that there is a constant $0 < \lambda \leq 1$, depending only on the quasisymmetry function of f, so that the balls

$$B(F(\hat{x}_1), \lambda \operatorname{diam} F(\hat{B}_1)), \ldots, B(F(\hat{x}_N), \lambda \operatorname{diam} F(\hat{B}_N))$$

are disjoint, where $\hat{x}_j$ is the center of $\hat{B}_j$. Thus, by (8.7) and (8.6), we have

$$\begin{aligned}\int_U \rho_\epsilon^Q \, d\mu &\leq \sum_{j=1}^N \int_{\hat{B}_j} \rho_\epsilon^Q \, d\mu \leq C \sum_{j=1}^N (\operatorname{diam} F(\hat{B}_j))^Q \\ &\leq C \sum_{j=1}^N \mathcal{H}_Q(B(F(\hat{x}_j), \lambda \operatorname{diam} F(\hat{B}_j))) \\ &\leq C\mathcal{H}_Q(F(\tilde{U})) < \infty.\end{aligned}$$

The proposition follows. □

Theorem 8.8. *In the situation of Proposition 8.4, F belongs to the local Sobolev class $N^{1,Q}_{\mathrm{loc}}(X : V)$.*

Proof. We have shown above that the functions F_ϵ are elements of $N^{1,Q}_{\mathrm{loc}}(X : V)$ which converge locally uniformly to F and have V-upper gradients ρ_ϵ that are uniformly in $L^Q_{\mathrm{loc}}(X)$. Following [49], one can easily verify that $F \in N^{1,Q}_{\mathrm{loc}}(X : V)$; we briefly recall this argument. First, by Mazur's lemma, we may choose a suitable sequence of convex combinations of the functions ρ_ϵ (which we continue to denote by ρ_ϵ) that converge to a limit Borel function ρ in $L^Q_{\mathrm{loc}}(X)$. The corresponding sequence of convex combinations of the functions F_ϵ continues to converge locally uniformly to F. Fuglede's lemma 3.4 implies that for Q-almost every curve γ, we have $\int_\gamma \rho_\epsilon \, ds \to \int_\gamma \rho \, ds$ as $\epsilon \to 0$. We pass to the limit as $\epsilon \to 0$ in the inequality

$$\|F_\epsilon(x) - F_\epsilon(y)\| \leq C \int_\gamma \rho_\epsilon \, ds$$

to conclude that $\rho \in L^Q_{\mathrm{loc}}(X)$ is a Q-weak V-upper gradient of F. The theorem follows. □

Remark 8.9. Note that we do not know whether the pertinent Sobolev spaces are reflexive in the generality of Theorem 8.8. The argument in the proof does not assert that they are; the membership of F in $N^{1,Q}_{\mathrm{loc}}(X : V)$ is guaranteed by more direct means. Essentially this is possible by Fuglede's lemma and by the use of weak upper gradients in the definition of the Sobolev space. Recall that Cheeger [9] has shown reflexivity under the assumption that the underlying space supports the p-Poincaré inequality (4.2) for some $p < \infty$ and $V = \mathbb{R}$.

The hypotheses in Theorem 8.8 are rather mild. For example, they are valid for quasisymmetric embeddings $f : \mathbb{R}^n \to \mathbb{R}^N$, $2 \leq n \leq N$, provided $f(\mathbb{R}^n)$ has locally finite Hausdorff n-measure (thus, $Q = n$ here). The crucial property (8.6) was proven by Väisälä in [57]. Väisälä also proves that the coordinate functions of f belong to the standard Sobolev space $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$ in this case. It is interesting to compare the two proofs; the convolution argument in [57] is replaced here by a simple discrete method.

We next show that the hypotheses are valid in much more generality.

Recall from [24] that a path-connected metric measure space $X = (X, d, \mu)$ is called a *Q-Loewner space*, $Q > 1$, if its Loewner function $\phi_{X,Q} : (0, \infty) \to \mathbb{R}$ is everywhere positive. The *Loewner function of exponent Q* of an arbitrary metric measure space X is defined to be

$$\phi_{X,Q}(t) = \inf_{E,F} \operatorname{mod}_Q \Delta(E, F),$$

where the infimum is taken over all pairs of disjoint nondegenerate continua $E, F \subset X$ satisfying $\operatorname{dist}(E, F) \leq t \min\{\operatorname{diam} E, \operatorname{diam} F\}$. Here $\Delta(E, F)$ denotes the collection of all curves in X joining E to F and $\operatorname{mod}_Q \Gamma$ denotes the *Q-modulus* of the curve family Γ as defined in 3.1. See [24] or [52] for more discussion on the Loewner condition.

If X is a Q-Loewner space, then the Q-measure of balls in X satisfies a uniform lower mass bound: there exists a constant $C \geq 1$, depending only on the Loewner data of X, so that $\mathcal{H}_Q(B_R) \geq C^{-1} R^Q$ for all balls B_R in X of radius $0 < R < \operatorname{diam} X$. See [24, Section 3]. In fact, a similar bound holds for the Hausdorff Q-measure in each quasisymmetric image of a Q-Loewner space:

Proposition 8.10. *If F is an η-quasisymmetric embedding of a Q-Loewner space X in a Banach space V, then there exists a constant $C \geq 1$, depending only on η and the Loewner data of X, so that (8.6) holds for each ball B_R of radius* $0 < R < \operatorname{diam} F(X)$.

A proof for Proposition 8.10 can be found in [54, Theorem 3.4] in the case $X = \mathbb{R}^n$, $n \geq 2$. The case of a general Loewner space X is similar, see [54, Remark 3.5(1)].

We are now ready to state the main result of this section. Before this, recall that a metric space X is *Ahlfors Q-regular*, $Q > 0$, if there exists a constant $C \geq 1$ so that

$$\frac{1}{C} r^Q \leq \mathcal{H}_Q(B(x, r)) \leq C r^Q \tag{8.11}$$

for all balls $B(x, r)$ in X with $0 < r < \operatorname{diam} X$; such a space necessarily has Hausdorff dimension Q. Also, X is called *proper* if each closed ball in X is compact.

Theorem 8.12. *Let X be a proper and pathwise connected Ahlfors Q-regular metric space supporting the Q-Poincaré inequality, $Q > 1$. Assume that $F : X \to V$ is a quasisymmetric embedding of X in a Banach space V so that the Hausdorff Q-measure on $F(X)$ is locally finite. Then $F \in N^{1,Q}_{\mathrm{loc}}(X : V)$. Moreover, F is pseudomonotone, whence F is absolutely continuous in measure: if $E \subset X$ satisfies $\mathcal{H}_Q(E) = 0$, then $\mathcal{H}_Q(F(E)) = 0$.*

Theorem 8.12 follows from the preceding discussion and from other known results as follows. First, the hypotheses imply that X is a Q-Loewner space by [24, Theorem 5.7]. Here one needs the fact that X is quasiconvex (see [18, Proposition 4.4] or [9, Theorem 17.1] for a proof). By Proposition 8.10 and Theorem 8.8 we have that $F \in N^{1,Q}_{\mathrm{loc}}(X : V)$. The final assertion follows from Theorem 7.2. We only need the following lemma:

Lemma 8.13. *In the situation of Theorem 8.12, F is pseudomonotone.*

Proof. Let $R_0 < \frac{1}{4} \operatorname{diam} X$ and let $B(x,r)$ be a ball in X with radius $r < R_0$. Because $X \setminus B(x, 2r) \neq \emptyset$, it follows from the linear local connectivity of Q-regular Q-Loewner spaces [24, Theorem 3.13] that $\operatorname{diam} \partial B(x,r) > \lambda r$ for some fixed $\lambda > 0$ depending only on the data of the problem. Pick $y, z \in \partial B(x,r)$ so that $d(y,z) > \lambda r$. Then for $a \in B(x,r)$ we have that

$$\|F(y) - F(a)\| \leq \eta(2/\lambda)\|F(y) - F(z)\|$$

which implies that

$$\operatorname{diam} F(B(x,r)) \leq 2\eta(2/\lambda) \operatorname{diam} F(\partial B(x,r)) .$$

The lemma follows. □

As mentioned above, proper and pathwise connnected Ahlfors Q-regular spaces that support the Q-Poincaré inequality are Q-Loewner. There is also a converse to this result: every locally compact Q-regular Q-Loewner space supports the Q-Poincaré inequality, see [24, Theorem 5.12]. For completeness, we next state a result for Q-Loewner spaces.

Theorem 8.14. *Let X be a locally compact Ahlfors Q-regular Q-Loewner space, $Q > 1$. Assume that $F : X \to V$ is a quasisymmetric embedding of X in a Banach space V so that the Hausdorff Q-measure on $F(X)$ is locally finite. Then $F \in N^{1,Q}_{\mathrm{loc}}(X : V)$. Moreover, F is pseudomonotone, whence F is absolutely continuous in measure: if $E \subset X$ satisfies $\mathcal{H}_Q(E) = 0$, then $\mathcal{H}_Q(F(E)) = 0$.*

The Loewner condition with exponent $Q > 1$ is quasisymmetrically invariant in locally compact Q-regular spaces [52]. We thus have the following corollary:

Corollary 8.15. *Let X and Y be locally compact Ahlfors Q-regular spaces, $Q > 1$. Assume that X is a Q-Loewner space and let $F : X \to Y$ be a quasisymmetric homeomorphism. Then Y is a Q-Loewner space and both F and F^{-1} are absolutely continuous with respect to the Hausdorff Q-measure.*

Remark 8.16. Because the conclusions in Theorems 8.8 and 8.14, and in Corollary 8.15, are local, the hypotheses can be localized as well (we forgo precise formulations here for simplicity). The local versions will be used in the next section.

Historical remarks. The absolute continuity result of Theorem 8.12 in the classical setting of quasiconformal self-maps of the Euclidean space $\mathbb{R}^n$, $n \geq 2$, was first established by Gehring [14] and Väisälä [55]. In [16] and [57], absolute continuity in measure was established for locally quasisymmetric embeddings f of $\mathbb{R}^n$ into $\mathbb{R}^N$, $2 \leq n \leq N < \infty$, for which the

Hausdorff n-measure is locally finite in $f(\mathbb{R}^n)$. Recently, Tyson in [54] dispensed with the assumption that the image embeds in a Euclidean space. Pansu [42] and Koranyi-Reimann [29] considered the case of Carnot groups. A different generalization was given in [24], where Heinonen and Koskela gave an abstract formulation of the Gehring-Väisälä theorem in which $\mathbb{R}^n$ was replaced by an arbitrary Ahlfors Q-regular space, $Q > 1$, satisfying the p-Poincaré inequality for some $p < Q$; this latter assumption is quasisymmetrically invariant in Q-regular spaces by [32]. Theorems 8.12 and 8.14 contain all of these previously known results.

We now give an alternative proof of Corollary 8.15 which uses only the Loewner condition and not the Q-Poincaré inequality. We begin by recalling that $N^{1,Q}_{\mathrm{loc}}(X : V)$ maps are absolutely continuous with respect to arc length measure along Q-almost every curve. That is, the collection of rectifiable curves γ in X for which a map $F \in N^{1,Q}_{\mathrm{loc}}(X : V)$ is not absolutely continuous with respect to arc length measure along γ has Q-modulus zero.

We shall show that for quasisymmetric maps between Ahlfors Q-regular spaces absolute continuity along Q-almost every curve implies absolute continuity in measure. This latter result can be found in [53, Theorem 5.9]; we repeat it here for the sake of completeness.

For a Borel set $A \subset X$, let Γ^+_A denote the collection of locally rectifiable curves γ with positive length in A, i.e. $\int_\gamma \chi_A \, ds > 0$. Note that Γ^+_A has Q-modulus zero whenever $\mu(A) = 0$ since the function $\rho = \infty \cdot \chi_A$ is admissible and Q-integrable. It turns out that in Loewner spaces the converse holds. (Compare Theorem 33.1 of [56].)

Lemma 8.17. *Let $X = (X, d, \mu)$ be an Ahlfors Q-regular Q-Loewner space, $Q > 1$, and let A be a Borel subset of X for which Γ^+_A has Q-modulus zero. Then $\mu(A) = 0$.*

Lemma 8.17 follows by combining Theorem 6.2 and Lemma 6.5 of [49], see also [48, Proposition 4.2.13] or [53, Lemma 5.10]. Corollary 8.15 follows directly by using this lemma. Suppose that $F : X \to Y$ is as in the hypotheses of Corollary 8.15, and assume that $E' \subset Y$ satisfies $\mathcal{H}_Q(E') = 0$ but $\mathcal{H}_Q(F^{-1}(E')) > 0$. Then $\operatorname{mod}_Q \Gamma^+_{E'} = 0$ and Lemma 8.17 implies that

$$\operatorname{mod}_Q \Gamma^+_{F^{-1}(E')} > 0.$$

Since Q-exceptionality of a curve family is a quasisymmetric invariant of locally compact Q-regular spaces, $Q > 1$, see [52, Theorem 1.4], we see that

$$\operatorname{mod}_Q F^{-1}\Gamma^+_{E'} = 0.$$

But

$$\Gamma^+_{F^{-1}(E')} \subset F^{-1}\Gamma^+_{E'} \cup \Gamma_{NR} \cup \Gamma_{NAC}\,,$$

where Γ_{NR} denotes the collection of non-locally rectifiable curves in X and Γ_{NAC} denotes the collection of locally rectifiable curves γ in X such that F^{-1} is not absolutely continuous on $F \circ \gamma$. Finally, both Γ_{NR} and Γ_{NAC} have Q-modulus zero. This leads to a contradiction and thus the proof of Corollary 8.15 is complete.

9. Definitions for quasiconformality

In this section, we shall show that the three main definitions for quasiconformality in Euclidean space can similarly be used in a general setting of (locally) Loewner metric spaces. Here we rely on the results from the previous sections as well as on results from [24].

Definition 9.1. A metric measure space (X, μ) is said to be of *locally Q-bounded geometry*, $Q > 1$, if X is separable, pathwise connected, locally compact, and if there exist constants $C_0 \geq 1$, $0 < \lambda \leq 1$, and a decreasing function $\psi : (0, \infty) \to (0, \infty)$ so that the following holds: each point in X has a neighborhood U such that

$$\mu(B_R) \leq C_0 R^Q \tag{9.2}$$

whenever $B_R \subset U$ is a ball of radius $R > 0$, and that

$$\operatorname{mod}_Q(E, F; B_R) \geq \psi(t) \tag{9.3}$$

whenever $B_R \subset U$ is a ball of radius $R > 0$ and E and F are two disjoint, nondegenerate continua in $B_{\lambda R}$ with $\operatorname{dist}(E, F) \leq t \min\{\operatorname{diam} E, \operatorname{diam} F\}$. For technical reasons we assume, as we may, that U has compact closure in X.

Remarks 9.4. (a) It is not hard to see that (9.3) implies a local lower mass bound

$$\mu(B_R) \geq C_1^{-1} R^Q, \qquad B_R \subset U, \tag{9.5}$$

for some $C_1 \geq 1$ depending only on ψ and λ (see the proof of Theorem 3.6 in [24]). Thus, a pathwise connected, locally compact space is of locally Q-bounded geometry if and only if it is locally uniformly Ahlfors Q-regular and satisfies the Loewner condition (9.3) locally uniformly. A similar but stronger (less local) condition of Q-bounded geometry was given in [7].

(b) One can replace (9.3) by the apparently weaker condition

$$\operatorname{mod}_Q(E, F; X) \geq \psi(t), \tag{9.6}$$

where E and F are two disjoint continua in $B_R \subset U$ satisfying $\operatorname{dist}(E, F) \leq t \min\{\operatorname{diam} E, \operatorname{diam} F\}$. Indeed, arguing as in [24, Section 3] one deduces from the volume growth condition (9.2) that (9.6) implies (9.3).

(c) It is clear that connected open subsets of spaces of locally Q-bounded geometry again have locally Q-bounded geometry (with the same data).

(d) It is also clear that every Riemannian n-manifold is of locally n-bounded geometry. More generally, every metric space X that is locally uniformly bi-Lipschitz equivalent to a Euclidean n-ball is of locally n-bounded geometry (with the Hausdorff n-measure in X). More exotic examples, including examples with nonintegral dimension $Q > 1$, are given in [8], [19], [24, Section 6], [34] and [46].

We call a homeomorphism $F : (X, d) \to (X', d')$ between metric spaces *quasiconformal*, or *H-quasiconformal*, $H \geq 1$, if

$$\limsup_{r \to 0} \frac{\sup\{d'(F(x), F(y)) : d(x, y) \leq r\}}{\inf\{d'(F(x), F(y)) : d(x, y) \geq r\}} =: H(x) \leq H \tag{9.7}$$

for each $x \in X$.

The following is the main result of this section:

Theorem 9.8. *Let $F : X \to Y$ be a homeomorphism between metric spaces of locally Q-bounded geometry. Then the following four conditions are quantitatively equivalent:*

(1) *F is H-quasiconformal;*
(2) *F is locally η-quasisymmetric;*
(3) *$F \in N^{1,Q}_{\mathrm{loc}}(X : Y)$ and $\operatorname{Lip} F(x)^Q \leq K J_F(x)$ for a.e. $x \in X$;*

(4) *the relation*

$$\frac{1}{L}\operatorname{mod}_Q\Gamma \leq \operatorname{mod}_Q F(\Gamma) \leq L\operatorname{mod}_Q\Gamma$$

holds for each curve family Γ *in* X.

Moreover, if one (each) of these conditions hold, then F *is absolutely continuous in measure and absolutely continuous along* Q*-a.e. curve in* X*, and* F^{-1} *is also quasiconformal.*

In part (3) of this theorem, $\operatorname{Lip} F(x)$ is defined as in Example 3.15, and $J_F(x)$ denotes the volume derivative

$$J_F(x) = \limsup_{r\to 0}\frac{\nu(F(B(x,r)))}{\mu(B(x,r))}, \tag{9.9}$$

whre ν is the measure in Y and μ is the measure in X. (Note that the limsup in (9.9) can be replaced by lim for a.e. $x \in X$.) Also recall that $N^{1,Q}_{\mathrm{loc}}(X:Y)$, by definition, denotes the collection of maps $F \in N^{1,Q}_{\mathrm{loc}}(X:\ell^\infty(Y))$ satisfying $F(x)\in Y$ for Q-quasi-every $x \in X$ (see Section 3).

Condition (2) means that every point in X has a neighborhood where F is η-quasisymmetric as defined in Section 8. Finally, the modulus of a curve family, $\operatorname{mod}_Q\Gamma$, was defined in (3.3).

Proof of Theorem 9.8. (1) $\Rightarrow$ (2). This can be proved by localizing the argument in Section 4 of [24]. Note here that the assumptions for locally bounded goemetry imply uniform *local linear connectivity* in a neighborhood of each point; see Section 3 of [24].

(2) $\Rightarrow$ (1). This is trivial.

(2) $\Rightarrow$ (3). If $F: X \to Y$ is locally quasisymmetric, then it is in $N^{1,Q}_{\mathrm{loc}}(X:Y)$ by Theorem 8.8 (or rather, by its obvious local version). On the other hand, for each $x\in X$, we easily see by quasisymmetry together with local Q-regularity that

$$\limsup_{r\to 0}\sup_{\substack{y\\ d(x,y)\leq r}}\frac{d'(F(x),F(y))^Q}{r^Q} \leq C\limsup_{r\to 0}\frac{\nu(F(B(x,r)))}{\mu(B(x,r))}.$$

Thus $\operatorname{Lip} F(x)^Q \leq C J_F(x)$ as desired.

(3) $\Rightarrow$ (4). As an element in $N^{1,Q}_{\mathrm{loc}}(X:Y)$, F is absolutely continuous along Q-a.e. curve γ. This is obvious because there exists an upper gradient $\rho \in L^Q_{\mathrm{loc}}$ so that (3.10) holds (locally) with finite right hand side for Q-a.e. curve. Because F and F^{-1} are, in addition, absolutely continuous in measure by (the obvious local version of) Corollary 8.15, the required quasi-invariance of the Q-modulus follows by standard methods. See, e.g., [24, Section 7].

(4) $\Rightarrow$ (1). This is a standard argument using the (uniform) local Loewner property together with the (uniform) local linear connectivity guaranteed by the hypotheses. See, for example, [24, Section 4] or page 79 of [56]. $\square$

Historical remarks. The history of various definitions for quasiconformal mappings is long. The mappings considered by Grötzsch and Teichmüller in the 1920's and 1930's were smooth. Ahlfors [1], in the third volume of *J. Analyse Math.* in 1954, made the first systematic study of nonsmooth quasiconformal mappings in dimension $n = 2$. Gehring [13] was the first to prove that the *metric* definition (9.7) in $\mathbb{R}^2$ implies quasiconformality according to the *analytic* 9.8(3) and *geometric* 9.8(4) definitions. In Euclidean n-space, $n \geq 3$, the equivalence of 9.8(1)–(4) was proved by Gehring [14] and Väisälä [55]. One should note, however, that the concept of quasisymmetry, although implicit in the early works, was not precisely formulated before the 1980 paper of Tukia and Väisälä [51].

Mostow [39], [40] was the first to consider quasiconformal mappings in non-Riemannian settings, on the boundaries of rank one symmetric spaces. Various definitions for quasiconformal mappings on general *Carnot groups* were considered by Pansu [42], Korányi-Reimann [28], and the Novosibirsk school [59]. The equivalence of the definitions 9.8(1)-(4) on the Heisenberg groups was proved by Korányi and Reimann [29] and, for the metric definition, by Mostow [41]. The case of general Carnot groups was settled independently by Margulis-Mostow [37] and Heinonen-Koskela [22] (see also [20]). The abstract methods of [22] were extended in [24] to Ahlfors Q-regular spaces that support the p-Poincaré inequality for some $p < Q$; although not explicitly stated in [24], Theorem 9.8 in that setting can be derived from the results in [24].

In this paper, we have arrived at the borderline spaces, that is, spaces that are Ahlfors Q-regular and support the Q-Poincaré inequality for some $Q > 1$.[3] One should note that Theorem 9.8 covers all of the cases mentioned in the previous paragraph. We suspect that the conditions in Theorem 9.8 are nearly weakest (although it is hard to make this precise) under which the equivalence of (1)-(4) is valid. However, the picture is nowhere near complete for some of the individual implications which can hold in greater generality. For example, it follows from [53] that 9.8(2) implies (4) for homeomorphisms between arbitrary locally compact Ahlfors Q-regular spaces (regardless of whether the spaces admit rectifiable curves or not). Also, the equivalence of 9.8(1) and (2) was shown in [21] to hold for homeomorphisms between compact polyhedra; these polyhedra need not be Ahlfors regular nor need they support the p-Poincaré inequality for p below or at the Hausdorff dimension. Indeed, it is an interesting open question under what circumstances the two purely metric conditions 9.8(1) and (2) are equivalent. Even in $\mathbb{R}^n$, $n \geq 2$, there is no purely metric proof known for the fact that quasiconformality implies quasisymmetry. It is also not known whether, in $\mathbb{R}^n$, the distortion function η can be chosen to depend only on H and not on the dimension n. In particular, the problem in infinite dimensions is still open. See [58].

Next, we prove that quasiconformal mappings between metric spaces of locally Q-bounded geometry preserve the Sobolev space $N^{1,Q}$. This is a well-known result in the Euclidean (Riemannian) setting. See also [32].

Theorem 9.10. *Let $F : X \to Y$ be a quasiconformal homeomorphism between metric spaces of locally Q-bounded geometry. If $u \in N^{1,Q}_{\mathrm{loc}}(Y)$, then $u \circ F \in N^{1,Q}_{\mathrm{loc}}(X)$. Moreover, if $B \subset X$ is a ball with compact closure, then*

$$\int_B \rho_{u\circ F}^Q \, d\mu \leq C \int_{F(B)} \rho_u^Q \, d\nu \,, \tag{9.11}$$

[3]For the record, the case $Q = 1$ would be false: on the real line $\mathbb{R}$, the infinitesimal condition 9.8(1) does not imply local quasisymmetry (2), and quasisymmetric maps need not be absolutely continuous and hence in particular need not be members of $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$.

where $C \geq 1$ depends only on the constant of quasiconformality of F and the data associated with X and Y.

Here μ and ν denote the measures in X and Y, respectively, and $\rho_\star$ denotes the minimal weak upper gradient as defined in 3.1.

Proof. Let $u \in N^{1,Q}_{\mathrm{loc}}(Y)$, and let $B \subset X$ be a ball with compact closure $\overline{B}$ in X such that $u \in N^{1,Q}(\Omega)$ for some open neighborhood Ω of $F(\overline{B})$. Assume further that B is small enough so that the Q-Poincaré inequality holds in some neighborhoods of B and $F(B)$; it is convenient to think of these neighborhoods as large in comparison with B and $F(B)$. It suffices to show that $u \circ F \in L^Q(B)$ and that

$$\int_B \rho_{u\circ F}^Q \, d\mu \leq C \int_{F(B)} \rho_u^Q \, d\nu \,, \tag{9.12}$$

with $C \geq 1$ depending only on the data. Indeed, if B is an arbitrary ball in X with compact closure, then it can be covered by finitely many balls for which (9.12) holds, and these balls can be chosen to have bounded (depending only on the data) overlap, cf. Section 8.

The arguments in [48, Lemma 5.2.7], [47, Lemma 2.14] show that we can multiply u by a Lipschitz cut-off function φ so that $\varphi \equiv 1$ in a neighborhood of $F(\overline{B})$ and that φ has compact support in Ω; then $\varphi \cdot u \in N^{1,Q}(Y)$ and

$$\rho_{\varphi\cdot u}\big|_{F(B)} = \rho_u\big|_{F(B)} \,.$$

It follows that without loss of generality we may assume that $u \in N^{1,Q}(Y)$ with compact support, and that the Q-Poincaré inequality holds in all of Y. Thus, u can be approximated in $N^{1,Q}(Y)$ by Lipschitz functions with supports in a fixed compact set [47, Theorem 4.8]. Let (φ_j) be such a sequence. Now consider the functions $\varphi_j \circ F$. Because F is absolutely continuous along Q-a.e. curve in X by Theorem 9.8, it follows as in the proof of 3.13 that $\mathrm{Lip}(\varphi_j \circ F)$ is a Q-weak upper gradient of $\varphi_j \circ F$. On the other hand, for $x \in X$,

$$\mathrm{Lip}(\varphi_j \circ F)(x) \leq \mathrm{Lip}\,\varphi_j(F(x))\,\mathrm{Lip}\,F(x) \leq C\,\mathrm{Lip}\,\varphi_j(F(x))\,J_F(x)^{1/Q}$$

so that

$$\begin{aligned} \int_X \mathrm{Lip}(\varphi_j \circ F)(x)^Q \, d\mu(x) &\leq C \int_X \mathrm{Lip}\,\varphi_j(F(x))^Q J_F(x)\, d\mu(x) \\ &= C \int_Y \mathrm{Lip}\,\varphi_j(y)^Q \, d\nu(y). \end{aligned} \tag{9.13}$$

The last equality is valid because F is absolutely continuous in measure. We next invoke a result of Cheeger [9, Proposition 4.2], according to which

$$\mathrm{Lip}\,\varphi_j(y) \leq C \rho_{\varphi_j}(y) \tag{9.14}$$

for a.e. $y \in Y$, where $C \geq 1$ depends only on the data associated with Y. Therefore, because $\varphi_j \to u$ in $N^{1,Q}(Y)$, we have a uniform bound (independent of j) for the integral on the left in (9.13). Similarly, we claim that

$$\sup_j \int_X |(\varphi_j \circ F)(x)|^Q \, d\mu(x) < \infty. \tag{9.15}$$

Indeed, because the functions $\varphi_j \circ F$ have supports in a fixed compact set in X, and because X supports (locally) the Q-Poincaré inequality with (locally) a doubling measure μ, it follows from [18] that

$$\int_X |\varphi_j \circ F|^Q \, d\mu \leq C \int_X \operatorname{Lip}(\varphi_j \circ F)^Q \, d\mu \,,$$

and (9.15) follows from (9.13) and the remarks following it.

We have now shown that $\varphi_j \circ F$ is in $N^{1,Q}(X)$ with a uniform bound for the norm:

$$\sup_j \|\varphi_j \circ F\|_{1,Q} < \infty.$$

It now follows from [49, Lemma 4.11] that there is a function $v \in N^{1,Q}(X)$ so that

$$\int_X |v|^Q \, d\mu \leq \liminf_{k\to\infty} \int_X |\varphi_{j_k} \circ F|^Q \, d\mu \tag{9.16}$$

and

$$\int_X \rho_v^Q \, d\mu \leq \liminf_{k\to\infty} \int_X \rho^Q_{\varphi_{j_k}\circ F} \, d\mu$$

for some subsequence (φ_{j_k}) of (φ_j).[4] Indeed, v is the weak L^Q-limit of the sequence $(\varphi_{j_k} \circ F)$; we may choose the subsequence so that $\varphi_{j_k} \circ F$ converges to $u \circ F$ pointwise a.e. Now standard arguments show that $v = u \circ F$ (note that the absolute continuity of F is needed here again).

This understood, we have that $u \circ F \in N^{1,Q}(X)$ and that

$$\int_X \rho_{u\circ F}(x)^Q \, d\mu(x) \leq C \liminf_{k\to\infty} \int_Y \operatorname{Lip} \varphi_{j_k}(y)^Q \, d\nu(y). \tag{9.17}$$

Upon observing that $\rho_{\varphi_j} \leq \rho_{u-\varphi_j} + \rho_u$ and that $\rho_{u-\varphi_j}$ tends to zero in $L^Q(Y)$, we obtain from (9.17) and (9.14) that

$$\int_X \rho^Q_{u\circ F} \, d\mu \leq C \int_Y \rho_u^Q \, d\nu$$

as desired. The proof of Theorem 9.10 is now complete. □

Remarks 9.18. Removable sets for quasiconformal mappings and Sobolev functions in abstract settings have been studied in [4] and [33], respectively. We next point out how some of the results about mappings in [4] can be reduced, via Theorem 9.8, to the general results found in [33].

Let X and Y be proper, pathwise connected, Ahlfors Q-regular metric spaces supporting the Q-Poincaré inequality; in this setting this is equivalent to saying that X and Y are both Q-Loewner. (Recall that a proper space is one where closed balls are compact.) Let $E \subset X$ be a closed set of measure zero. We say that E is *removable for quasiconformal mappings* if each homeomorphism $F : X \to Y$ which is quasiconformal in $X \setminus E$ (in the infinitesimal sense of (9.7)) is in fact quasiconformal in all of X. (Note that we assume *a priori* that $F : X \to Y$ is a homeomorphism; it is equally relevant but a somewhat different problem to study removability for maps that are only defined in $X \setminus E$.)

In the present setting, we have that $F \in N^{1,Q}_{\mathrm{loc}}(X \setminus E)$ by Theorem 9.8. Thus F will be quasiconformal in all of X if we know that E is *removable for continuous Sobolev functions* in the following sense: for each $x \in E$ there is $r > 0$ so that every continuous function

[4] Note that these facts would also follow from Cheeger's reflexivity theorem [9], upon noting the equivalence of Cheeger's Sobolev space with $N^{1,p}$, but the argument based on weak upper gradients in [49] is much simpler and does not use reflexivity.

defined in the ball $B(x,r)$ whose restriction to $B(x,r) \setminus E$ lies in $N^{1,Q}(B(x,r) \setminus E)$ is in fact an element of $N^{1,Q}(B(x,r))$. This understood, it follows that the removability result [4, Theorem 1.2] for quasiconformal mappings is a consequence of the removability result [33, Theorem B] for continuous Sobolev functions. It was not clear before this how (if at all) these two results are connected.

Finally, we mention that the problem of characterizing removable sets for quasiconformal mappings is not fully understood even for homeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$. In the abstract setting, the only known results are those in [4] and [33]. Although the removable sets described in those papers can be quite large, it is not known in what generality a rectifiable curve, say, is removable for $N^{1,Q}$ functions in an Ahlfors Q-regular space supporting the Q-Poincaré inequality. This is the case in $\mathbb{R}^n$, $n \geq 2$, and in some other nice situations. The general case remains open.

10. Cheeger differentials

In a recent paper [9], Cheeger shows how to obtain an analog of the classical theorem of Rademacher on differentiability a.e. of Lipschitz functions in Euclidean space on a doubling metric measure space X that supports the p-Poincaré inequality for some $1 < p < \infty$. In particular, he constructs a finite-dimensional L^∞ vector bundle $\mathbf{F}$ over X, certain sections of which correspond to the differentials of Lipschitz functions. The natural norm on the fibers of this vector bundle typically does not arise from an inner product and hence the resulting structure does not in general correspond to a Riemannian structure. Moreover, the "coordinate charts" are only measurable subsets of X and the transition functions only L^∞. Nevertheless, Cheeger's construction does allow him to formulate a theory of differentiability for a suitable class of maps between two such metric measure spaces which includes the case of quasisymmetric mappings between Ahlfors Q-regular spaces supporting the p-Poincaré inequality for $1 < p < Q$ [9]. In this section, we indicate how the results we have developed earlier in this paper extend to cover the borderline case $p = Q$.

We give first a slight restatement of one of the versions of the Rademacher theorem in Cheeger's paper, Theorem 4.38 of [9]. In what follows, by the *data* of the doubling metric measure space (X, d, μ) we mean the doubling constant of μ together with the constants C and σ associated with the Poincaré inequality (4.2). We denote by $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$ the space of linear functionals on $\mathbb{R}^k$ and we denote the action of an element $\lambda \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$ on a vector $v \in \mathbb{R}^k$ by $\langle \lambda, v \rangle$. Recall also the definition for a minimal p-weak upper gradient ρ_f from 3.1.

Theorem 10.1 (Cheeger). *Let $X = (X, d, \mu)$ be a doubling metric measure space supporting the p-Poincaré inequality for some $1 < p < \infty$. Then*

$$X = \bigcup_\alpha U_\alpha \cup Z,$$

where $\mu(U_\alpha) > 0$ for all α, $\mu(Z) = 0$, and to each α there correspond real-valued Lipschitz functions $x_1^\alpha, \ldots, x_k^\alpha$ on U_α satisfying the following three properties:

(i) *The functions $x_1^\alpha, \ldots, x_k^\alpha$ are linearly independent on U_α, i.e. if $\mathbf{X}^\alpha = (x_1^\alpha, \ldots, x_k^\alpha) : U_\alpha \to \mathbb{R}^k$, then for each $\lambda \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$ we have $\langle \lambda, \mathbf{X}^\alpha \rangle \equiv 0$ on U_α if and only if $\lambda = 0$;*

(ii) *for each $\lambda \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$ and $x_0 \in U_\alpha$, the function $\langle \lambda, \mathbf{X}^\alpha \rangle$ is asymptotically p-harmonic at x_0 (Definition 3.1 of [9]) and x_0 is a Lebesgue point of the function $\rho_{\langle \lambda, \mathbf{X}^\alpha \rangle}{}^p$; moreover, if $\lambda \neq 0$ then $\rho_{\langle \lambda, \mathbf{X}^\alpha \rangle}(x_0) > 0$;*

(iii) *k is the maximal integer for which (i) and (ii) hold, and $k = k(\alpha) \le N$, where $N \ge 1$ is a finite integer depending only on the data.*

Let $f : X \to \mathbb{R}$ be a Lipschitz function. Then, for each α, there corresponds to almost every point $x_0 \in U_\alpha$ a (unique) linear functional $\lambda = \lambda(x_0, f, \alpha) \in \mathrm{Hom}(\mathbb{R}^{k(\alpha)}, \mathbb{R})$ so that

$$\rho_{-f+\langle\lambda,\mathbf{X}^\alpha\rangle}(x_0) = \lim_{\substack{x\to x_0\\ x\in U_\alpha}} \frac{|f(x) - f(x_0) - \langle\lambda, \mathbf{X}^\alpha(x) - \mathbf{X}^\alpha(x_0)\rangle|}{d(x,x_0)} = 0. \tag{10.2}$$

We write $d^\alpha f(x_0) := \lambda(x_0, f, \alpha)$. For fixed α, the map $x \mapsto d^\alpha f(x)$ is in $L^\infty(U_\alpha, \mu : \mathrm{Hom}(\mathbb{R}^{k(\alpha)}, \mathbb{R}))$. At a.e. point $x_0 \in U_\alpha$, we have

$$\rho_f(x_0) = \rho_{\langle d^\alpha f(x_0), \mathbf{X}^\alpha\rangle}(x_0). \tag{10.3}$$

The statement in (iii) requires some clarification. If V is a subset of X of positive μ-measure, define $k(V)$ to be the supremum of the values k for which there exist real-valued Lipschitz functions $\varphi_1, \ldots, \varphi_k$ on V satisfying (i) and (ii). Clearly, if W is a subset of V that also has positive μ-measure, then $k(W) \ge k(V)$ (just restrict the functions $\varphi_1, \ldots, \varphi_k$ to W). Let us say that V is *saturated* if $k(W) = k(V)$ for all such sets W. Then (iii) can be restated as saying that each of the sets U_α is saturated. This part of the theorem is easy to guarantee. Indeed, suppose we have found a decomposition $X = \cup_\alpha U_\alpha \cup Z$ satisfying (i) and (ii). If any of the sets U_α is not saturated, we may decompose it further into subsets on which the value of $k(\alpha)$ is increased by (at least) one. Lemma 4.37 of [9] gives an *a priori* finite upper bound for the values $k(\alpha)$ that may arise and shows that this process must terminate.

Roughly speaking, (10.3) says that the "derivative" of the map f at x_0 agrees with the "derivative" of its first-order Taylor approximation $\langle d^\alpha f(x_0), \mathbf{X}^\alpha\rangle$.

Note that we have restricted in (10.2) to $x \in U_\alpha$ since the function $\mathbf{X}^\alpha$ is *a priori* only defined on U_α. However, it is not hard to see that if we extend $\mathbf{X}^\alpha$ to a Lipschitz map from X to $\mathbb{R}$, then the limit in (10.2) is still zero if we allow x to approach x_0 throughout X. Moreover, this fact is independent of what extension of $\mathbf{X}^\alpha$ we use.

We now introduce, at each point $x_0 \in U_\alpha$, a norm $|\cdot|_{\alpha,x_0}$ on $\mathrm{Hom}(\mathbb{R}^{k(\alpha)}, \mathbb{R})$ as follows:

$$|\lambda|_{\alpha,x_0} = \rho_{\langle\lambda,\mathbf{X}^\alpha\rangle}(x_0).$$

The fact that this is a norm follows from Theorem 10.1(ii). By (10.3), we have $\rho_f(x_0) = |d^\alpha f(x_0)|_{\alpha,x_0}$. The Banach spaces

$$F_{x_0} = (\mathrm{Hom}(\mathbb{R}^{k(\alpha)}, \mathbb{R}), |\cdot|_{\alpha,x_0}), \qquad x_0 \in U_\alpha,$$

combine to give the finite-dimensional vector bundle $\mathbf{F}$ mentioned above. We call this bundle the *generalized cotangent bundle* of X and denote it by T^*X. By using local sections $d^\alpha f$ on U^α, one can define a derivation operator d on the algebra of locally Lipschitz functions on X which takes values in $\Gamma(\mathbf{F})$ (the sections of the vector bundle $\mathbf{F}$). We call df the *Cheeger differential* of the locally Lipschitz map f.

Now suppose that $X = (X, d, \mu)$ is as in Theorem 10.1 and that $Y = (Y, d')$ is another metric space (for now, we make no further assumptions on Y). If $F : X \to Y$ and $f : Y \to \mathbb{R}$ are Lipschitz functions, then $f \circ F : X \to \mathbb{R}$ is also Lipschitz and so by Theorem 10.1 the differentials $d^\alpha(f \circ F)$ are defined on sets U_α which cover μ-almost all of X. If now Y is given a measure ν so that (Y, d', ν) is also doubling and satisfies the p-Poincaré inequality, then the differentials $d^\beta f$ are defined on sets V_β covering ν-almost all of Y. At a.e. point

$x_0 \in F^{-1}(V_\beta) \cap U_\alpha$, there exists a unique linear map

$$D_F^{\alpha\beta}(x_0)^T : \mathrm{Hom}(\mathbb{R}^{k(\beta)}, \mathbb{R}) \to \mathrm{Hom}(\mathbb{R}^{k(\alpha)}, \mathbb{R}),$$

called the (transposed) *Jacobian matrix* of F at x_0, which satisfies the relation

$$D_F^{\alpha\beta}(x_0)^T \cdot d^\beta f(y_0) = d^\alpha (f \circ F)(x_0), \tag{10.4}$$

where $y_0 = F(x_0)$. The existence and uniqueness of this mapping are obvious. Indeed, taking $f = x_j^\beta$, $j = 1, \dots, k(\beta)$, in (10.4) and using the canonical identification of $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$ with $\mathbb{R}^k$ we see that the jth column of the $(k(\alpha) \times k(\beta))$ matrix representing $D_F^{\alpha\beta}(x_0)^T$ is given by the vector in $\mathbb{R}^{k(\alpha)}$ corresponding to $d^\alpha(x_j^\beta \circ F)(x_0)$.

Note, however, that the transposed Jacobian matrix is only defined on the set

$$\bigcup_{\alpha,\beta} F^{-1}(V_\beta) \cap U_\alpha$$

in X. We would like to guarantee that this is a set of full measure in X but in general this is not true. In order to ensure that this is the case, we must further assume that $\nu(A) = 0$ implies $\mu(F^{-1}A) = 0$, that is, that the push-forward measure $F_*\mu$ is absolutely continuous with respect to ν. When this is the case, we have a natural induced map $F^* : T^*Y \to T^*X$ satisfying

$$F^*(df) = d(f \circ F), \tag{10.5}$$

where d is the Cheeger differential. When expressed in coordinate charts $U_\alpha \subset X$ and $V_\beta \subset Y$, F^* is just the transposed Jacobian matrix $D_F^{\alpha\beta}(x_0)^T$ and (10.5) becomes (10.4).

Now assume that $X = (X, d, \mathcal{H}_Q)$ and $Y = (Y, d', \mathcal{H}_Q)$ are proper and path-connected Ahlfors Q-regular metric measure spaces, $Q > 1$, both supporting the Q-Poincaré inequality, and assume that $F : X \to Y$ is a quasiconformal homeomorphism. Then X is a Q-Loewner space and so the push-forward measure $F_*\mathcal{H}_Q$ is absolutely continuous with respect to $\mathcal{H}_Q$ by Corollary 8.15 and by [24]. Moreover, $F \in N^{1,Q}_{\mathrm{loc}}(X : V)$, where $V = \ell^\infty(Y)$. If ρ denotes a Q-weak V-upper gradient of F in $L^Q_{\mathrm{loc}}(X)$, then Lemma 4.6 implies that, locally in X,

$$\|F(x) - F(y)\| \le C d(x,y)(M_{2\sigma d(x,y),Q}\rho(x) + M_{2\sigma d(x,y),Q}\rho(y)), \tag{10.6}$$

where $M_{R,p}$ is the maximal function operator defined in (4.5). For $\lambda > 0$ and for U a small enough bounded neighborhood of a given point in X, define

$$E_\lambda(U) := \{x \in U : M_{4\sigma \operatorname{diam} U, Q}\rho(x) \le \lambda\}.$$

Then F is Lipschitz with constant $C\lambda$ when restricted to the set $E_\lambda(U)$ by (10.6).

Lemma 10.7. *Under the above assumptions,*

$$\mu(U \setminus E_\lambda(U)) \le C\lambda^{-Q} \int_{\tilde U} \rho^Q \, d\mu,$$

where $\tilde U = \{x \in X : \operatorname{dist}(x, U) < 4\sigma \operatorname{diam} U\}$.

This lemma is nothing more than the boundedness of the Hardy-Littlewood maximal function from $L^1(X)$ to weak $-L^1(X)$, see e.g. [18, Theorem 14.13].

By choosing a sequence $\lambda_j \to \infty$ and a sequence of bounded sets U_k that satisfy the conclusion of Lemma 10.7 and cover almost all of X, we find that the collection of sets $E_{jk} := E_{\lambda_j}(U_k)$ covers μ-almost all of X. If $f : Y \to \mathbb{R}$ is any Lipschitz function, then $f \circ F$ is Lipschitz when restricted to any of the sets E_{jk} and the previous discussion is valid. In conclusion, we deduce the following result:

Theorem 10.8. *Let X and Y be proper and path-connected Ahlfors Q-regular spaces, $Q > 1$, supporting the Q-Poincaré inequality, and let $F : X \to Y$ be a quasiconformal homeomorphism. Then there is a natural induced map $F^* : T^*Y \to T^*X$ satisfying*

$$F^*(df) = d(f \circ F),$$

for all Lipschitz functions $f : Y \to \mathbb{R}$, where d is the Cheeger differential.

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