Existence and Uniqueness Theorems for First-Order ODE's

The general first-order ODE is

$$y' = F(x, y), \qquad y(x_0) = y_0. \tag{*}$$

We are interested in the following questions:
(i) Under what conditions can we be sure that a solution to (*) exists?
(ii) Under what conditions can we be sure that there is a unique solution to (*)?

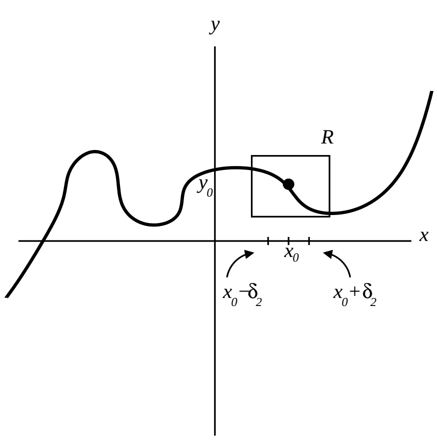
Here are the answers.

Theorem 1 (Existence). *Suppose that $F(x, y)$ is a continuous function defined in some region*

$$R = \{(x, y) : x_0 - \delta < x < x_0 + \delta, y_0 - \epsilon < y < y_0 + \epsilon\}$$

containing the point (x_0, y_0). Then there exists a number δ_1 (possibly smaller than δ) so that a solution $y = f(x)$ to (*) is defined for $x_0 - \delta_1 < x < x_0 + \delta_1$.

Theorem 2 (Uniqueness). *Suppose that both $F(x, y)$ and $\frac{\partial F}{\partial y}(x, y)$ are continuous functions defined on a region R as in Theorem 1. Then there exists a number δ_2 (possibly smaller than δ_1) so that the solution $y = f(x)$ to (*), whose existence was guaranteed by Theorem 1, is the unique solution to (*) for $x_0 - \delta_2 < x < x_0 + \delta_2$.*



For a real number x and a positive value δ, the set of numbers x satisfying $x_0 - \delta < x < x_0 + \delta$ is called an *open interval* centered at x_0.

Example 3. Consider the ODE

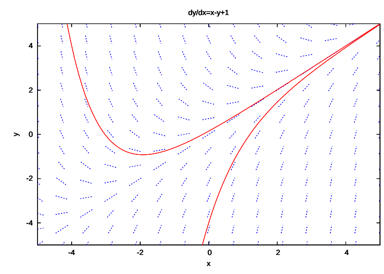
$$y' = x - y + 1, \qquad y(1) = 2.$$

In this case, both the function $F(x, y) = x - y + 1$ and its partial derivative $\frac{\partial F}{\partial y}(x, y) = -1$ are defined and continuous at all points (x, y). The theorem guarantees that a solution to the ODE exists in some open interval centered at 1, and that this solution is unique in some (possibly smaller) interval centered at 1.

In fact, an explicit solution to this equation is

$$y(x) = x + e^{1-x}.$$

(Check this for yourself.) This solution exists (and is the unique solution to the equation) for all real numbers x. In other words, in this example we may choose the numbers δ_1 and δ_2 as large as we please.



Example 4. Consider the ODE

$$y' = 1 + y^2, \qquad y(0) = 0.$$

Again, both $F(x, y) = 1 + y^2$ and $\frac{\partial F}{\partial y}(x, y) = 2y$ are defined and continuous at all points (x, y), so by the theorem we can conclude that a solution exists in some open interval centered at 0, and is unique in some (possibly smaller) interval centered at 0.

By separating variables and integrating, we derive a solution to this equation of the form

$$y(x) = \tan(x).$$

As an abstract function of x, this is defined for all $x \neq \ldots, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, \ldots$. However, in order for this function to be considered as a solution to this ODE, we must restrict the domain. (Remember that a solution to a differential equation must be a *continuous* function!) Specifically, the function

$$y = \tan(x), \qquad -\pi/2 < x < \pi/2,$$

is a solution to the above ODE.

In this example we must choose $\delta_1 = \delta_2 = \pi/2$, although the initial value δ may be chosen as large as we please.

Example 5. Consider the ODE

$$y' = 2y/x, \qquad y(x_0) = y_0.$$

In this example, $F(x, y) = 2y/x$ and $\frac{\partial F}{\partial y}(x, y) = 2/x$. Both of these functions are defined for all $x \neq 0$, so Theorem 2 tells us that for each $x_0 \neq 0$ there exists a unique solution defined in an open interval around x_0.

By separating variables and integrating, we derive solutions to this equation of the form

$$y(x) = Cx^2$$

for any constant C. Notice that all of these solutions pass through the point $(0, 0)$, and that none of them pass through any point $(0, y_0)$ with $y_0 \neq 0$. So the initial value problem

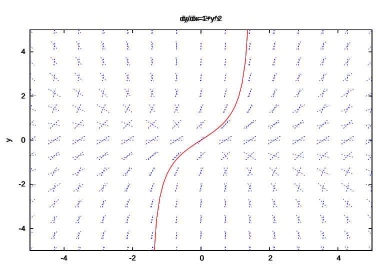
$$y' = 2y/x, \qquad y(0) = 0,$$

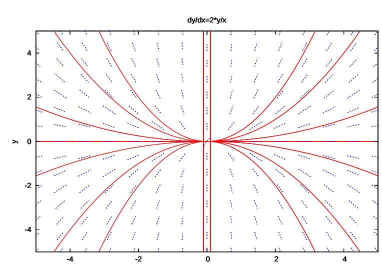
has infinitely many solutions, but the initial value problem

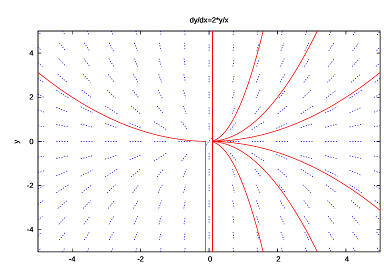
$$y' = 2y/x, \qquad y(0) = y_0,\ y_0 \neq 0,$$

has no solutions.

For each (x_0, y_0) with $x_0 \neq 0$, there is a unique parabola $y = Cx^2$ whose graph passes through (x_0, y_0). (Choose $C = y_0/x_0^2$.) So the initial value problem $y' = 2y/x$, $y(x_0) = y_0$, $x_0 \neq 0$, has a unique solution defined on some interval centered at the point x_0. In fact, in this case, there exists a solution which is defined for all values of x (δ_1 may be chosen as large as we please), but that there is a unique solution only on the interval $x_0 - \delta_2 < x < x_0 + \delta_2$, where $\delta_2 = |x_0|$. This examples shows that the values δ_1 and δ_2 may be different.







Summary. The initial value problem $y(x_0) = y_0$ has

- a unique solution in an open interval containing x_0 if $x_0 \neq 0$;
- no solution if $x_0 = 0$ and $y_0 \neq 0$;
- infinitely many solutions if $(x_0, y_0) = (0, 0)$.