Quasiconformal geometry of boundaries of hyperbolic spaces

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AMS Special Session on Geometry of Nilpotent Groups
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Slides available at

http://www.math.uiuc.edu/~tyson/bowdoin.pdf

or

https://sites.google.com/site/lucacapogna/meetings-talks

Overview

This talk is about the geometry of negatively curved (hyperbolic) spaces and its relationship to analysis on their boundaries at infinity.

We focus on the **coarse** geometry of hyperbolic space. Quasi-isometric mappings of hyperbolic spaces act on the boundary as quasiconformal (QC) mappings. Quasiconformality measures uniform infinitesimal relative metric distortion. QC mappings can also be understood and studied from an analytic perspective.

Motivated by Mostow's proof of his rigidity theorem, we pay special attention to the classical rank one symmetric spaces, where the natural structure on the boundary at infinity is either Riemannian or sub-Riemannian. We will discuss several aspects of the theory of QC maps in sub-Riemannian manifolds, including equivalence of definitions, Liouville-type rigidity of 1-QC maps, and extension theorems.

We start with a brief review of quasiconformal mapping theory in Euclidean space.

Quasiconformal mappings: three definitions

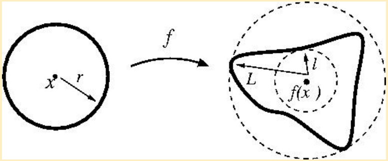
Roughly speaking, a quasiconformal map is a homeomorphism for which the infinitesimal relative distortion of distance is uniformly bounded.

Definition

Let $f : X \to Y$ be a homeomorphism between metric spaces (X, d) and (Y, d'). We say that f is *(metrically) H-quasiconformal (H-QC)* for some $H \geq 1$ if

$$\limsup_{r \to 0} \frac{L(x, f, r)}{\ell(x, f, r)} \leq H$$

for all $x \in X$.



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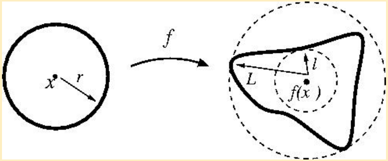


E.g., conformal maps between planar domains are 1-QC. Much of QC mapping theory consists in understanding how geometric and analytic properties of conformal maps (in the setting of complex analysis) generalize to higher dimensions and to (nonsmooth) metric spaces.

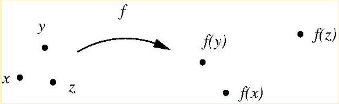
Quasiconformal mappings: three definitions

Metric QC is an infinitesimal condition which is usually too weak to work with effectively. It is more convenient to work with a global distortion condition.

Definition

Let $f : X \to Y$ be a homeomorphism between metric spaces (X, d) and (Y, d'), and let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. We say that f is η-*quasisymmetric* if

$$\frac{d'(f(x), f(y))}{d'(f(x), f(z))} \le \eta\left(\frac{d(x, y)}{d(x, z)}\right)$$



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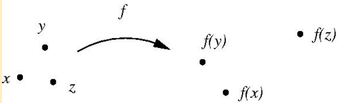


for distinct $x, y, z \in X$.

E.g., every **bi-Lipschitz map** is QS (f L-BL $\Rightarrow f$ η-QS, $\eta(t) = L^2 t$).

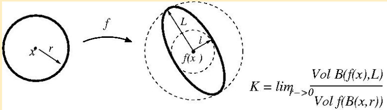
$f : X \to Y$ is a **snowflake mapping** if there exists $0 < \epsilon \leq 1$ s.t. f is (L-)BL from X to (Y, d_Y^ϵ). Snowflake maps are QS ($\eta(t) = L^2 t^\epsilon$).

Quasiconformal mappings: three definitions

Definition

Let $f : \Omega \to \Omega'$ be a homeo between domains in $\mathbb{R}^n$, $n \geq 2$. We say that f is *(analytically) K-QC* if f lies in the local Sobolev space $W^{1,n}_{loc}$ and

$$||Df||^n \leq K \det Df \text{ a.e.} \quad (1)$$



(1) asserts that the local length distortion induced by f is consistent with the local volume distortion. In particular, it implies that the maximal and minimal length distortion[1] induced by f are comparable, with a comparison constant that is uniformly bounded over all points in the domain.

[1] At points of differentiability of f, these are the largest and smallest singular values of Df.

Quasiconformal mappings: three definitions

Since length is defined in terms of the Euclidean (inner product) norm on tangent spaces, it is not surprising that (1) can be restated in terms of distortion of the Riemannian structure. Define a bounded measurable function G from Ω to the space $S(n)$ of symmetric positive definite matrices in $\mathrm{SL}_n(\mathbb{R})$:

$$Df(x)^T Df(x) = (\det Df(x))^{2/n} G(x) \,. \qquad (2)$$

(2) is known as the **Beltrami system**, and G is the **distortion tensor**. The special case $G(x) = \mathbf{I}_n$ is the **Cauchy–Riemann system**

$$Df(x)^T Df(x) = (\det Df(x))^{2/n} \mathbf{I}_n \,, \qquad (3)$$

which is satisfied by 1-QC maps.

Quasiconformal mappings: a brief history

- Grötzsch (1928): extremal problems in complex analysis
- 1930–1960: planar theory (Teichmüller, Ahlfors, Bers, Beurling, Lehto, ...)
  connections to Teichmüller theory/Riemann surfaces/ quadratic differentials, univalent function theory

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- 1960–1980s: *n* dimensional and Riemannian theory (Gehring, Väisälä, Mostow, Donaldson–Sullivan)
  Mostow rigidity theorem, Kleinian groups, complex dynamics, differential geometry (Donaldson–Sullivan theory)

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Mostow rigidity theorem, Kleinian groups, complex dynamics, differential geometry (Donaldson–Sullivan theory)

- 1980s–present: QC mappings in non-Riemannian metric spaces (Pansu, Korányi–Reimann, Heinonen, Koskela, . . .)
geometric group theory, sub-Riemannian geometry, first-order regularity theory for mappings between metric spaces

Quasiconformal maps: two fundamental theorems

Theorem (cf. Lehto–Virtanen, Gehring, Väisälä, . . .)

For homeomorphisms between domains in $\mathbb{R}^n$, $n \geq 2$,

metric QC $\Leftrightarrow$ local QS $\Leftrightarrow$ analytic QC

A key step in the proof is to show that metrically QC maps are **absolutely continuous along lines**: the restriction of the map to almost every line segment parallel to one of the coordinate axes is absolutely continuous.

The ACL property ensures that the pointwise differential exists almost everywhere, after which membership in the Sobolev class and the differential inequality follow easily.

In higher dimensions ($n \geq 3$) the ACL property for QC mappings is due to Gehring (1962).

Quasiconformal maps: two fundamental theorems

1-QC maps of planar domains are precisely the conformal (or anti-conformal) mappings. The Riemann mapping theorem ensures that there is a rich supply of such maps.

Theorem (Liouville 1850 ; Menchoff 1937 ; Gehring 1962 ; cf. also Reshetnyak, Iwaniec–Martin, . . .)

1-*QC mappings of domains in* $\mathbb{R}^n$, $n \geq 3$ *are conformal (i.e., restrictions of Möbius transformations).*

The first step in the proof is to show that 1-QC maps are smooth (C^∞). One approach is to prove that 1-QC maps preserve the class of n-**harmonic functions**, i.e., solutions to $\sum_j \partial_j(|\nabla u|^{n-2}\partial_j u) = 0$.

Since the coordinate functions in $\mathbb{R}^n$ are n-harmonic, it follows that the components of a 1-QC map f are n-harmonic.

Elliptic regularity theory ensures that n-harmonic functions whose gradient is bounded away from zero and infinity are smooth.

Euclidean quasiconformal maps: extension theorems and connections to hyperbolic geometry

Theorem (Beurling–Ahlfors 1956)

Every QC self-map of $U := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ *extends as a QS self-map of* $\mathbb{R}$*. Conversely, every QS self-map of* $\mathbb{R}$ *admits a QC extension to a self-mapping of* U*.*

Theorem (Ahlfors ; Carleson ; Tukia–Väisälä 1982)

Every QC self-map of $\mathbb{R}^n$ *extends as a QC self-map of* $\mathbb{R}^{n+1}$*. More precisely, if* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is QC, then* $\exists\, F : H^{n+1}_{\mathbb{R}} \to H^{n+1}_{\mathbb{R}}$ *bi-Lipschitz s.t.* $F|_{\partial H^{n+1}_{\mathbb{R}} = \mathbb{R}^n} = f$*.*

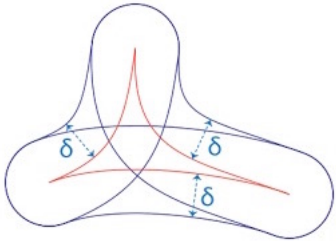
Real hyperbolic space $H^{n+1}_{\mathbb{R}}$ is conformally equivalent to the upper half space $U = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$. So f extends to a QC self-map of U, and then extends to a QC self-map of $\mathbb{R}^{n+1}$ by reflection.

Gromov hyperbolic spaces

Gromov hyperbolic spaces yield a far-reaching generalization of the discussion on the previous slide. Set $\delta \geq 0$ to be a sort of *thin-ness* parameter or hyperbolicity constant.

Hyperbolicity

A geodesic metric space (X, d) is $\delta-$hyperbolic if for every geodesic triangle, any side lies in the $\delta-$neighborhood of the other two sides.



Gromov hyperbolic spaces

Definition

A metric space (X, d) is $\delta-$hyperbolic if for all $x, y, z, o \in X$ one has

$$(x, y)_o \geq \min\Big((x, z)_o, (z, y)_o\Big) - \delta$$

Here we have denoted by $(x, y)_o$ the *Gromov* product of x, y with respect to the basepoint o

$$(x, y)_o = \frac{1}{2}\Big(d(o, x) + d(o, y) - d(x, y)\Big)$$

Buyalo-Schroeder:

"Roughly speaking, in a $\delta-$hyperbolic space X, two sides $\bar{xy}$, $\bar{xz}$ of a geodesic triangle with vertices x, y, z, run together within distance δ up to length $(y, z)_x$ and after that they start to diverge."

Visual boundary

Fix a basepoint $o \in X$. A sequence $\{x_i\}$ *converges to infinity* if

$$\lim_{m,n\to\infty}(x_m, x_n)_o = \infty$$

Two sequences $\{x_m\}$ and $\{y_n\}$ converging to infinity are equivalent if

$$\lim_{n\to\infty}(x_n, y_n)_o = \infty$$

This is independent from the choice of basepoint o.

Definition

The visual boundary $\partial_G X$ is the set of equivalence classes of sequences converging to infinity.

Given $a, b \in \partial_G X$, we set

$$(a, b)_o = \sup \lim_{n\to\infty}(x_n, y_n)_o$$

where the sup is taken over all representatives a, b.

Visual metrics

For a fixed base point $o \in X$ and for every $\epsilon > 0$ one can define a quasi-metric $d_{\epsilon,o}$ on $\partial_G X$ in the following way

$$d_{\epsilon,o}(a,b) = \exp(-\epsilon(a,b)_o)$$

Visual Metric

For all $o \in X$, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, $d_{\epsilon,o}$ is bi-Lipschitz to a distance function on $\partial_G X$. Such metrics are called visual metrics.

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- Any two visual metrics corresponding to the same parameter ϵ but different base points are bi-Lipschitz.
- Any two visual metrics corresponding to different parameters ϵ, ϵ' are snowflake equivalent. In particular, they are quasisymmetrically equivalent.

Examples of Gromov hyperbolic spaces

- Real hyperbolic space This is $\delta-$hyperbolic for $\delta < \ln 3$. The visual boundary can be identified with the unit sphere or an hyperplane (depending on which model one chooses) and the visual metric corresponding to $\epsilon = 1$ can be computed explicitly, resulting in the spherical distance or the Euclidean metric.
- Hadamard manifolds with sectional curvature $K \leq -a^2$ These are $\delta-$hyperbolic for the same δ as the hyperbolicity parameter of the space form with constant sectional curvature $-a^2$.
- Complex hyperbolic space Here the visual boundary is the Heisenberg group and the induced visual metric for $\epsilon = 1$ is the Korányi gauge.
- Symmetric spaces of rank one, with constant negative curvature The visual boundary is a Heisenberg type (H-type) group, and the visual metric corresponding to $\epsilon = 1$ is a gauge quasi-norm.

These are all examples of the class

- CAT(-1) spaces

More examples of Gromov hyperbolic spaces

Examples of Gromov hyperbolic spaces that are not CAT(-1) can be obtained by compact perturbations of a CAT(-1) space or for instance from the following theorem due to Balogh and Bonk:

- Strictly Pseudo-Convex domains in $\mathbb{C}^n$ are Gromov hyperbolic with respect to the Bergman metric or to the Kobayashi Finsler metric. The visual boundary coincides with the topological boundary and the visual metric corresponding to $\epsilon = 1$ is the sub-Riemannian distance function associated to the Levi form.

We also mention that the Cayley graph associated to a finite generating set for a group G is Gromov hyperbolic with respect to the word metric for a *generic* finitely presented group. Specific examples include the fundamental group of the sphere with two handles and the free group F_n of rank n. (see the beautiful survey of Kapovich and Benakli).

Quasi-isometries and boundary extensions

The dual relation that exists for mappings between Gromov hyperbolic spaces and their visual boundary provides a common context for problems coming from different parts of mathematics. The fundamental concept that emerges is that all asymptotic properties of a hyperbolic space are encoded in its boundary at infinity.

Definition

Let $f : X \to Y$ be a map (with cobounded range) between metric spaces and let $k \geq 0$ and $\lambda_2 \geq 1 \geq \lambda_1$. If for all $x, y \in X$ one has

$$\lambda_1 d(x,y) - k \leq d(f(x), f(y)) \leq \lambda_2 d(x,y) + k$$

Then f is called a $(\lambda, k)-$rough quasiisometry. If $\lambda_1 = \lambda_2 = 1$ then f is called a rough isometry.

k yields additive noise - roughness

Quasi-isometries and boundary extensions

The following theorem is due to many authors (See for instance the work of Bonk and Schramm or the monograph of Buyalo-Schroeder for references)

Theorem

Rough isometries between Gromov hyperbolic spaces induce bi-Lipschitz maps between the visual boundaries. Viceversa, any bi-Lipschitz embedding between boundaries (with visual distances corresponding to the same parameter ϵ) can be extended to a rough isometry between the spaces.

More generally ...

rough isometries induce bi-Lipschitz maps

rough similarities induce snowflake maps

rough quasiisometries induce power quasi-symmetries

A result of Bonk and Schramm

Theorem

Let X be a Gromov hyperbolic geodesic space. If the boundary $\partial_G X$ is doubling for some visual metric, then X is roughly similar to a convex subset of a real hyperbolic space of dimension $n \geq 2$.

Consider also the work of Bonk, Heinonen and Koskela who have proved that (roughly speaking) every (proper, geodesic, roughly star-like) Gromov hyperbolic space arises as a conformal image of a bounded uniform space. This correspondence turns (in a conformal fashion) the unbounded structure of a Gromov hyperbolic space into a bounded space with nice internal geometry. This provides a dictionary to translate properties of bounded Euclidean domains into properties of Gromov hyperbolic spaces.

An application that ties together all these ideas: Mostow rigidity

One of the deepest applications of the tight connection between maps between Gromov hyperbolic spaces and maps between their visual boundaries is Mostow's proof of his rigidity theorem.

Mostow rigidity I: "Algebraic" version

Let G be the group of isometries of hyperbolic space $H^{n+1}_{\mathbb{R}}$. Let Γ, Γ' be subgroups such that $\Gamma \backslash G$ and $\Gamma' \backslash G$ have finite Haar measure. Let $\theta : \Gamma \to \Gamma'$ be an isomorphism, $f : H^{n+1}_{\mathbb{R}} \to H^{n+1}_{\mathbb{R}}$ a quasiconformal map, such that

$$f(\gamma x) = \theta(\gamma) f(x) \text{ for all } \gamma \in \Gamma \text{ and } x \in H^{n+1}_{\mathbb{R}}$$

If $n > 1$ then θ extends to an inner automorphism of G.

Mostow rigidity

Mostow rigidity: A 'geometric' take

If M, N are compact hyperbolic manifolds of dimension three or larger, with isomorphic fundamental groups, then they are isometric. Moreover the isomorphism is induced by a unique isometry.

Mostow rigidity: Two immediate consequences

Corollary

If two finite volume, complete Riemannian manifolds with constant negative curvature are quasiconformally equivalent, and their dimension is ≥ 3, then they are also conformally equivalent. Moreover there is a unique conformal map that induces the same isomorphism between the fundamental groups as a given quasiconformal relation.

Corollary

If two compact Riemannian manifolds with same constant negative curvature are diffeomorphic, and their dimension is three or larger, then they are isometric.

This is false in dimension two! Riemann surfaces with negative constant curvature need not be conformally equivalent (this is the beginning of Teichmüller theory).

Strategy of the proof

- Either show that the group isomorphism θ extends to a quasi-isometry $\psi : H_{\mathbb{R}}^{n+1} \to H_{\mathbb{R}}^{n+1}$, or start directly with a given ψ.

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- Extend the boundary conformal correspondence to an isometry that is $\Gamma-$invariant.

Mostow's rigidity II

Let S_1, S_2 be rank-one symmetric spaces (different from the hyperbolic 2-space). Consider two lattices Γ_1, Γ_2 in the groups of isometries of S_1, S_2. Any group isomorphism $\phi : \Gamma_1 \to \Gamma_2$ arises as the conjugacy of an isometry between S_1 and S_2.

The proof follows the outline from the previous slide but now there are new issues to be tackled, since the geometry of the visual boundary is no longer locally Euclidean.

Symmetric spaces of rank one and their visual boundaries

There are only four types of non-compact, rank one symmetric spaces. Namely, the hyperbolic spaces over the real numbers, the complex numbers, the quaternions, and one exceptional space, the two-dimensional hyperbolic space over the Cayley numbers. (see for instance the work of Cowling, Dooley, Koranyi and Ricci)

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In all such cases the visual boundary can be identified as a Lie group, with a nilpotent Lie algebra of step 2, endowed with a particular visual distance, the Carnot–Carathéodory distance, which does not arise out of any Riemannian or Finsler metric, but is rather a sub-Riemannian object.

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Morever the groups in question possess a particular algebraic structure, they are H(eisenberg)-type groups or Kaplan groups which we describe next.

Algebraic structure of H-type groups

H-type groups were introduced by Kaplan in 1980. An H-type group is a Lie group G, with a two step, nilpotent Lie algebra $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$. endowed with a scalar product $\langle \cdot, \cdot \rangle$ (with corresponding norm $|\cdot|$) for which $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal, and a map

$$J : \mathfrak{z} \to End(\mathfrak{v})$$

satisfying

$$\langle J(Z)X, Y \rangle = \langle Z, [X, Y] \rangle \text{ for any } X, Y \in \mathfrak{v} \text{ and } Z \in \mathfrak{z}$$

and

$$|J(Z)X| = |Z||X|$$

Vectors in $\mathfrak{v}$ are called horizontal.

sub-Riemannian geometry of H-type groups

The Carnot–Carathéodory distance d in G, associated to this algebraic H-structure, is defined as

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The metric space so obtained can be seen as a Gromov–Hausdorff limit of Riemannian metric spaces, whose curvature explodes at every point along vertical directions.

sub-Riemannian geometry of H-type groups

- Every two points have a geodesic connecting them, but it may not be unique.
- The injectivity radius is zero.
- Hausdorff dimension is $Q = \dim(\mathfrak{v}) + 2\dim(\mathfrak{z})$, so distinct from topological dimension.
- smooth curves are not rectifiable unless they are horizontal
- smooth diffeomorphisms may not be bi-Lipschitz
- natural heat/laplace operators are not elliptic

Two aspects of the theory of sub-Riemannian QC maps

Developing the strategy of the proof of Mostow rigidity in this broader, non Riemannian, context has led to the study of sub-Riemannian quasiconformal mapping theory.

As is characteristic when dealing with quasiconformal mappings, their study both leads to and requires new and substantial progress in different areas of mathematics. Here we want to recount how the study of sub-Riemannian quasiconformal mappings provided motivation and model problems for

- the development of analysis in metric spaces, and
- the development of certain non-linear subellliptic Partial Differential Equations and the associated Potential Theory.

sub-Riemannian QC mapping theory and the origin of analysis in metric spaces

Following Mostow's work, a comprehensive study of QC maps in the Heisenberg group was undertaken by Korányi and Reimann (1985 ; 1995). In particular, they studied Heisenberg QC maps according to various definitions (metric, analytic, geometric), developed a theory of quasiconformal flows, and proved a version of the Liouville theorem on 1-QC maps.

Recall that a key step in Gehring's original proof of the implication "metric quasiconformality $\Rightarrow$ analytic quasiconformality" in the Euclidean setting was the fact that metrically QC maps are ACL (absolutely continuous on lines). Put another way, such maps are absolutely continuous along almost every fiber of any orthogonal projection mapping to a hyperplane. The proof of this fact turns crucially on the observation that projection mappings are Lipschitz.

sub-Riemannian QC mapping theory and the origin of analysis in metric spaces

During the preparation of their 1995 'Foundations' paper on Heisenberg QC mappings, Korányi and Reimann observed that the corresponding projection-type mappings of the Heisenberg group, whose fibers determine the natural foliation w.r.t. which the ACL property is understood, are **not** Lipschitz.

sub-Riemannian QC mapping theory and the origin of analysis in metric spaces

G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton Univ. Press, 1973.

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sub-Riemannian QC mapping theory and the origin of analysis in metric spaces

"...the crucial ACL regularity condition for quasiconformal mappings cannot be proved as easily as in the Euclidean case. Mostow had overlooked this difficulty in his original proof of the ACL regularity. But once we brought this point to his attention, he worked out a complete proof ..."

A. Korányi and H.-M. Reimann, 'Foundations for the theory of QC mappings on the Heisenberg group', *Adv. Math.*, 1995

sub-Riemannian QC mapping theory and the origin of analysis in metric spaces

The search for an alternate route to derive quasisymmetry from metric quasiconformality (avoiding any analytic discussion) led Heinonen and Koskela to introduce the concept of **Poincaré inequality** for a metric measure space. Eventually, this led to a fully-fledged theory of first-order analysis and geometry, including quasiconformal mapping theory, in doubling metric measure spaces supporting a Poincaré inequality.

We will have more to say about these developments later in the talk. For now, we turn to a second line of new research associated with sub-Riemannian QC mapping theory, connected to subelliptic PDE.

sub-Riemannian QC mapping theory and nonlinear subelliptic PDE

At the core of the proof of Mostow rigidity theorem, in the classical case, there is the following statement:

Lemma

Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a 1-quasiconformal map. If $n \geq 2$ then f is a Möbius transformation.

Mostow proof relies on ergodicity of the group action. It can be generalized to the case of boundaries of symmetric spaces, but works only for globally defined maps. At the same time, the classical Liouville theorem (described earlier) applies also to locally defined maps.

Rigidity of local 1-quasiconformal maps: PDE enter the picture

Korányi and Reimann were the first to prove a sub-Riemannian analogue of the Liouville theorem.

Theorem

Let $f : \Omega \to \Omega'$ be a $1-$quasiconformal map between open domains of the Heisenberg group. If f is C^4, then f is the action of a group element of $SU(1,2)$ restricted to Ω.

As in the classical Liouville theorem, the proof is essentially based on the fact that the Beltrami system is overdetermined, and then relies on theorems from several complex variables.

Rigidity of local 1-quasiconformal maps: PDE enter the picture

Lowering the regularity assumption on the map requires new ideas. The method described earlier, for the Euclidean case, relies on the study of the PDE that arises as Euler-Lagrange equation of the conformal energy, i.e. the subelliptic $Q-$Laplacian

$$L_Q : u = \sum_{i=1}^{2n} X_i^* (|\nabla_H u|^{Q-2} X_i u) = 0,$$

here Q is the Hausdorff dimension of the sub-Riemannian Heisenberg group, and $X_1, ..., X_{2n}$ are an orthonormal frame of horizontal vector fields.

The degenerate elliptic character of this PDE arises from the fact that only horizontal derivatives are present and from the possible blow-up or vanishing of the horizontal gradient.

Regularity of solutions of the subelliptic $Q-$Laplacian and similar quasilinear degenerate elliptic PDE: a partial biblio

- Harnack inequalities and Holder regularity of solutions (1992) Biroli, Mosco, Capogna, Danielli, Garofalo, Lu (most of this work, and similar results fall into the Saloff-Coste/Grygorian frame)
- Regularity of the gradient for equations with linear growth in the Heisenberg group (1996), Capogna.
- Regularity of the gradient for equations and systems with linear growth in Carnot groups (1999) and ongoing; Capogna, Garofalo, Foglein, Shores, Bramanti, Xu, Lu, ...
- Regularity of the gradient for more general growth, but for partial range of $2 \leq p < 4$ in the Heisenberg group, (2003) Marchi, (2004) Domokos; (2005) Domokos and Manfredi; Manfredi and Mingione (2006); Mingione, Goldstein and Zhong (2007).
- Regularity of the gradient for more general growth in the Heisenberg group, (2008) Zhong (extended to contact manifolds in 2016 by Capogna, Le Donne and Ottazzi)

Rigidity of local $1-$quasiconformal maps: PDE enter the picture

The sub-Riemannian Liouville theorem is tightly connected to regularity for the $Q-$Laplace equation. In fact, one has the following

Theorem

For every sub-Riemannian manifold, If $Q-$harmonic functions have Hölder continuous horizontal derivatives then locally defined $1-$quasiconformal maps are smooth. In particular the result holds for sub-Riemannian contact manifolds.

Using the additional structure of nilpotent Lie groups, Capogna and Cowling (2006) have proved the smoothness of $1-$quasiconformal mappings for Carnot groups, generalizing earlier work of Tang (1996).

Note that the sharp regularity of $Q-$harmonic functions is not known for Carnot groups.

This finishes our discussion of the regularity theory for subelliptic PDE as it relates to the Liouville theorem in sub-Riemannian geometry and Mostow rigidity.

In the remainder of the talk, we will circle back to and expand on several topics mentioned earlier:

- extension theorems for quasisymmetric maps on the boundaries of Gromov hyperbolic spaces,
- quasiconformal maps in metric measure spaces: equivalence of definitions,
- rigidity of quasiconformal mappings in sub-Riemannian spaces.

Extension theorems

Recall that every quasisymmetric map $f : \partial_\infty X \to \partial_\infty Y$ between boundaries of Gromov hyperbolic spaces admits a quasi-isometric extension $F : X \to Y$ of the interiors.

When $X = Y = H^{n+1}_{\mathbb{R}}$ and $\partial_\infty X = \partial_\infty Y = \mathbb{S}^n$, every QS map $f : \partial_\infty X \to \partial_\infty Y$ admits a bi-Lipschitz extension $F : X \to Y$ (Beurling–Ahlfors, Carleson, Tukia–Väisälä).

Another recent extension theorem due to Lemm and Markovic resolves the Schoen conjecture in the real hyperbolic case.

Theorem (Markovic ; Lemm–Markovic)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a QS map. Then f admits an extension $F : H^{n+1}_{\mathbb{R}} \to H^{n+1}_{\mathbb{R}}$ which is harmonic and quasi-isometric.

Equivalently, every quasi-isometry of $H^{n+1}_{\mathbb{R}}$ is at bounded distance from a harmonic quasi-isometry.

Extension theorems

Pansu (1989) proved that **every** quasiconformal mapping of the quaternionic or Cayley Heisenberg groups is necessarily 1-quasiconformal. Phrased in the language of the hyperbolic interior, this says that *every quasi-isometry of quaternionic hyperbolic space or the Cayley hyperbolic plane is at bounded distance from an isometry.* Pansu's result provided part of the original motivation for the Schoen conjecture.

Benoist and Hulin have recently announced the complete solution to the Schoen conjecture for rank one symmetric spaces by resolving the case of complex hyperbolic space. Namely, they show that every QS map of the Heisenberg group $\mathbb{H}^n$ admits an extension to $H_{\mathbb{C}}^{n+1}$ which is harmonic and quasi-isometric.

A Tukia–Väisälä extension thm for rank one symm spaces

Alternatively, in the spirit of the Tukia–Väisälä extension theorem, one may ask whether every QS map of the boundary of a rank one symmetric space admits a bi-Lipschitz extension.

The following result of Xiangdong Xie implies such a conclusion in all cases except that of the complex hyperbolic plane $H^2_{\mathbb{C}}$ and its boundary at infinity (locally modeled on the first Heisenberg group $\mathbb{H}^1$).

Theorem (Xie, 2009)

Let X and Y be the universal covers of two compact Riemannian manifolds of negative sectional curvature. Assume that neither X nor Y is 4-dimensional. Then every quasi-isometry from X to Y lies at a bounded distance from a bi-Lipschitz map from X to Y.

A Tukia–Väisälä extension thm for rank one symm spaces

Lukyanenko (2013) proved a similar QS to BL extension theorem for a class of metric spaces arising as boundaries at infinity of Gromov hyperbolic Riemannian manifolds. These manifolds may have positive curvature in places, and need not admit co-compact isometric group actions. Lukyanenko's result also covers the case of rank one symmetric spaces and their boundaries. There is a similar restriction to dimensions $\neq 4$, so again the complex hyperbolic plane $H^2_{\mathbb{C}}$ is ruled out.

Question

Does every QS map $f : \mathbb{H}^1 \to \mathbb{H}^1$ admit a bi-Lipschitz extension $F : H^2_{\mathbb{C}} \to H^2_{\mathbb{C}}$?

The dimension restriction (in both cases) is due to the existence of exotic Lipschitz structures on 4-manifolds and related issues in connection with Sullivan's Approximation Theorem.

Quasiconformal mapping theory in metric measure spaces: further developments

Heinonen and Koskela (1998) introduced the notion of Poincaré inequality for a metric measure space (X, d, μ). This axiom is satisfied by Euclidean space and all sub-Riemannian Carnot groups, e.g., the Heisenberg group.

The fundamental underlying concept is the notion of upper gradient, which is a metric space abstraction of the Euclidean norm of the classical gradient of a C^1 function.

In her thesis (1999), Shanmugalingam defined the first-order Sobolev space $N^{1,p}(X, d, \mu)$ (Newtonian–Sobolev space) to be the class of $L^p(X, \mu)$ functions which admit an L^p upper gradient. By standard methods (isometric embedding into Banach spaces), one can then make sense of the class of Sobolev mappings between metric measure spaces: $N^{1,p}(X : Y)$.

Theorem (Heinonen–Koskela–Shanmugalingam–T, 2001)

Let $f : X \to Y$ be a homeomorphism between metric measure spaces (X, d, μ) and (Y, d', ν). Assume that (X, d, μ) and (Y, d', ν) are Ahlfors Q-regular and satisfy the Q-Poincaré inequality for some $Q > 1$. Then the following are equivalent:

- *f is metrically quasiconformal,*
- *f is quasisymmetric,*
- *f lies in the local Sobolev space $N^{1,Q}_{loc}(X : Y)$ and the ineq*

$$(\mathrm{Lip}\, f)(x)^Q \le K J_f(x)$$

holds for μ-a.e. $x \in X$, for some fixed $K \ge 1$.

$$(\mathrm{Lip}\, f)(x) := \limsup_{y\to x} \frac{d'(fx, fy)}{d(x,y)} \text{ and } J_f(x) = \frac{d(f_\# \nu)}{d\mu}(x) = \lim_{r\to 0} \frac{\nu(fB(x,r))}{\mu(B(x,r))}$$

denote the maximal stretch factor and volume derivative respectively.

Quasiconformal mapping theory in metric measure spaces: further developments

Research into the equivalence of definitions of quasiconformality in the metric measure space setting is ongoing. In particular, individual implications among the various definitions are known to hold under weaker assumptions on the source and target.

Results of this type in either sub-Riemannian or metric spaces have been established by Tyson, Williams, Rajala and others. The complete story is not yet known.

Quasiconformal mappings of Carnot groups: rigidity

The preceding result applies in particular to mappings between Carnot groups, and indicates that all standard definitions for quasiconformality continue to coincide in this setting.

The Korányi–Reimann theory demonstrates that the class of QC maps of the Heisenberg group is rich, for instance, there are nontrivial one-parameter deformations of mappings (a QC flow theory). This has been effectively demonstrated in the construction of nonsmooth QC maps, starting with the 'liftings' of planar non-smooth maps (Capogna, Tang, 1995) all the way to the construction of QC maps that distort Hausdorff dimensions of subsets of $\mathbb{H}^n$ (Balogh, Tyson, Wildrick, 2013).

Quasiconformal mappings of Carnot groups: rigidity

On the other hand, Pansu's rigidity theorem asserts that all quasiconformal mappings of the remaining H-type groups arising in the setting of Mostow's theorem are necessarily conformal.

Extensive later work by many authors (Cowling, Warhurst, Ottazzi, Le Donne, Xie, ...) has revealed that such rigidity results are common. Indeed, one can assert that, roughly speaking, the complex Heisenberg group $\mathbb{H}^n$ provides one of the few genuinely sub-Riemannian Carnot groups in which a nontrivial quasiconformal mapping theory exists.

For more information, see tomorrow's talk by Xiangdong Xie.

References for further reading

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Available at

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