0. Introduction

Geometric measure theory considers the structure of Borel sets and Borel measures in metric spaces. It lies at the border between differential geometry and topology, and services a variety of areas: partial differential equations and the calculus of variations, geometric function theory, number theory, etc. The emphasis is on sets with a fine, irregular structure which cannot be well described by the classical tools of geometric analysis. Mandelbrot introduced the term “fractal” to describe sets of this nature. Dynamical systems provide a rich source of examples: Julia sets for rational maps of one complex variable, limit sets of Kleinian groups, attractors of iterated function systems and nonlinear differential systems, and so on.

In addition to its intrinsic interest, geometric measure theory has been a valuable tool for problems arising from real and complex analysis, harmonic analysis, PDE, and other fields. For instance, rectifiability criteria and metric curvature conditions played a key role in Tolsa’s resolution of the longstanding Painlevé problem on removable sets for bounded analytic functions.

Major topics within geometric measure theory which we will discuss include Hausdorff measure and dimension, density theorems, energy and capacity methods, almost sure dimension distortion theorems, Sobolev spaces, tangent measures, and rectifiability.

Rectifiable sets and measures provide a rich measure-theoretic generalization of smooth differential submanifolds and their volume measures. The theory of rectifiable sets can be viewed as an extension of differential geometry in which the basic machinery and tools of the subject (tangent spaces, differential operators, vector bundles) are replaced by approximate, measure-theoretic analogs. In contrast with smooth manifolds, rectifiable sets form a complete system; limits of rectifiable sets (in a suitable sense) are rectifiable.

Classically, geometric measure theory was developed in the setting of finite-dimensional Euclidean spaces. Contemporary trends in analysis and geometry in metric measure spaces, including analysis on fractals, necessitate extensions to non-Riemannian and nonsmooth spaces. Sub-Riemannian spaces, particularly, the sub-Riemannian Heisenberg group, are an important testing ground and model for the general theory.

In the first part of the course we will give an extended introduction to geometric measure theory in Euclidean spaces. In the second part we will discuss ongoing efforts to extend this theory into non-Riemannian and general metric spaces. Motivation for such efforts will be indicated. We will emphasize notions of rectifiability and almost sure dimension distortion theorems in sub-Riemannian spaces.

1. Review of Measure Theory

1.1. Metric spaces. A metric space is a set $X$ together with a nonnegative, symmetric function $d$ on $X \times X$ which satisfies the triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$$

and vanishes only on the diagonal of $X \times X$. We write $(X, d)$ for the space $X$ equipped with the metric $d$. Often we will not explicitly identify the metric $d$ under consideration. Our standard example is the Euclidean space $\mathbb{R}^n$ equipped with the Euclidean metric $d_E$:

$$d_E(x, y) = |x - y|, \quad |(x_1, \ldots, x_n)| = \sqrt{x_1^2 + \cdots + x_n^2}.$$ 

The open ball in $X$ with center $x \in X$ and radius $r \in (0, \infty)$ is

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$ 

The closed ball with this center and radius is

$$\overline{B}(x, r) = \{y \in X : d(x, y) \leq r\}.$$ 

Every metric space $(X, d)$ is equipped with a canonical topology, obtained by taking the family of open balls as a base.

In contrast with the situation in Euclidean space,

- the center and radius of a ball in a general metric space are not uniquely determined.
- the (topological) closure $\overline{B}(x, r)$ of $B(x, r)$ need not coincide with $\overline{B}(x, r)$.

The diameter of a (nonempty) subset $A \subset X$ is

$$\text{diam}(A) := \sup \{d(x, y) : x, y \in A\}$$

and the distance between two (nonempty) sets $A, B \subset X$ is

$$\text{dist}(A, B) = \inf \{d(x, y) : x \in A, y \in B\}.$$ 

We abbreviate $\text{dist}(\{x\}, B) = \text{dist}(x, B)$. Observe that $\text{diam}(A) = \text{diam}(\overline{A})$ and $\text{dist}(A, B) = \text{dist}(\overline{A}, \overline{B})$ for any $A, B \subset X$, where $\overline{A}$ denotes the closure of $A$. The diameter of any ball in a metric space satisfies $\text{diam} \overline{B}(x, r) \leq 2r$. Equality holds in $\mathbb{R}^n$ for all $x$ and all $r > 0$, but need not hold in general metric spaces.

For $\epsilon > 0$ and $A \subset X$ the $\epsilon$-neighborhood of $A$ in $X$ is the set

$$N_\epsilon(A) := \{x \in X : \text{dist}(x, A) < \epsilon\} = \bigcup_{x \in A} B(x, \epsilon).$$

1.2. Measures. For us, a measure on a space $X$ will mean a nonnegative (possibly infinite-valued), monotonic, countably subadditive set function defined on the power set of $X$ which vanishes on the empty set. More precisely, it is a function $\mu : \mathcal{P}(X) \to [0, \infty]$ (where $\mathcal{P}(X) = \{A : A \subset X\}$ denotes the power set of $X$) such that

$$\mu(\emptyset) = 0,$$

$$\mu(A) \leq \mu(B) \quad \text{if } A \subset B \subset X,$$

and

$$\mu\left(\bigcup_i A_i\right) \leq \sum_i \mu(A_i)$$

if $A_1, A_2, \cdots \subset X$. 

Recall that a set $A \subset X$ is called $(\mu)$-measurable if

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A) \quad \forall E \subset X.$$  

In other words, $A$ splits the measure of each subset of $E$. The collection $\mathcal{M}_\mu(X)$ of $\mu$-measurable sets forms a $\sigma$-algebra: it is closed under complements and countable unions. Each measure is countably additive on pairwise disjoint measurable sets: if $A_1, A_2, \ldots \in \mathcal{M}_\mu(X)$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then

$$\mu(\bigcup_i A_i) = \sum_i \mu(A_i). \quad (1.2.1)$$

**Remark 1.2.2.** What we call a measure here is what is usually called an “outer measure”, with the term “measure” reserved for the restriction of the outer measure to the $\sigma$-algebra of measurable sets. The terminology which we use here is traditional in geometric measure theory. It is essentially equivalent; any countable additive nonnegative set function defined on a $\sigma$-algebra of subsets of $X$ can be extended to a measure on $X$ (in the above sense). See Exercise 1.1 in [M].

Among various properties of measures and measurable sets we highlight the following.

**Proposition 1.2.3.** Let $A_1, A_2, \ldots \in \mathcal{M}_\mu(X)$.

(i) If $A_1 \subset A_2 \subset A_3 \subset \cdots$, then $\mu(\bigcap_{i=1}^\infty A_i) = \lim_{i \to \infty} \mu(A_i)$.

(ii) If $A_1 \supset A_2 \supset A_3 \supset \cdots$, and $\mu(A_1) < \infty$, then $\mu(\bigcap_{i=1}^\infty A_i) = \lim_{i \to \infty} \mu(A_i)$.

From now on we assume that the underlying space $X$ is equipped with a metric $d$. The Borel $\sigma$-algebra of $X$ is the smallest $\sigma$-algebra containing the open sets. Elements of the Borel $\sigma$-algebra are called **Borel sets**.

A measure $\mu$ on $(X, d)$ is said to be

- *finite* if $\mu(X) < \infty$;
- *locally finite* if for every $x \in X$ there exists $r > 0$ so that $\mu(B(x, r)) < \infty$;
- *$\sigma$-finite* if there exist $A_1, A_2, \ldots \subset X$ so that $X = \bigcup_{i=1}^\infty A_i$ and $\mu(A_i) < \infty$ for all $i$;
- *Borel* if every Borel set is measurable;
- *Borel regular* if it is Borel and each set $A$ is a subset of a Borel set $B$ with $\mu(A) = \mu(B)$;
- *Radon* if $\mu(K) < \infty$ for all compact sets $K$,

$$\mu(V) = \sup\{\mu(K) : K \subset V \text{ compact}\}$$

for all open sets $V$, and

$$\mu(A) = \inf\{\mu(V) : V \supset A \text{ open}\}$$

for all sets $A$;

- *doubling* if it is locally finite, Borel, and there exists a constant $C_\mu < \infty$ so that

$$0 < \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty$$

for all $x \in X$ and $r > 0$.

All of the measures we will consider in this course will be Borel, and virtually all of them will be Borel regular. Although many measures which we will consider will be locally finite, a notable exception is the family of Hausdorff measures which will figure prominently in the course (see subsection 1.4 for the definition). The Hausdorff measure $\mathcal{H}^\alpha$ on $\mathbb{R}^n$ is not locally finite for any $0 \leq \alpha < n$. 

All locally finite Borel regular measures on complete, separable metric spaces are Radon. Doubling measures are extensively studied in harmonic and geometric analysis. We will consider them in more detail in section 2.

A necessary and sufficient condition for a measure $\mu$ on a metric space $(X,d)$ to be Borel is Carathéodory’s criterion:
\begin{equation}
\mu(A \cup B) = \mu(A) + \mu(B)
\end{equation}
whenever $A,B \subset X$ satisfy $\operatorname{dist}(A,B) > 0$.

The support $\operatorname{spt} \mu$ of a Borel measure $\mu$ is the complement of the largest open set whose measure is zero. In other words,
$$\operatorname{spt} \mu := \bigcap \{F : F \subset X \text{ is closed, } \mu(X \setminus F) = 0\}.$$ 

**Exercise 1.2.5.** Show that $x \in \operatorname{spt} \mu$ if and only if $\mu(B(x,r)) > 0$ for all $r > 0$.

It is time to discuss examples.

**Example 1.2.6** (Counting measure). Let $X$ be a set. For each subset $A \subset X$ let $\mu(A)$ be the cardinality of $A$. Then $\mu$ is a measure in $X$. It is known as counting measure. Whenever $X$ is equipped with a metric, counting measure is Borel regular. It is finite if and only if $X$ is a finite set. It is locally finite if and only if $X$ is a discrete set (every compact subset of $X$ is finite). It is $\sigma$-finite if and only if $X$ is countable.

**Example 1.2.7** (Dirac measures). Let $X$ be a set. For fixed $a \in X$, let $\mu(A) = 1$ if $a \in A$ and $\mu(A) = 0$ otherwise. Then $\mu$ is a finite measure. It is known as the Dirac measure (or point mass) at $a$, and is typically denoted $\delta_a$. It is a Radon measure whenever $X$ is equipped with a metric.

The following example demonstrates some of the subtlety associated to the notion of the support of a measure.

**Example 1.2.8.** Let $q_1, q_2, \ldots$ be an enumeration of the rational numbers and define
$$\mu := \sum_{i=1}^{\infty} 2^{-i} \delta_{q_i}.$$ 

Then $\mu$ is a finite measure on $\mathbb{R}$, and $\mu(\mathbb{R} \setminus \mathbb{Q}) = 0$. However, $\operatorname{spt} \mu = \mathbb{R}$.

**Example 1.2.9** (Lebesgue measure). The Lebesgue measure $\mathcal{L}^n$ in the Euclidean space $\mathbb{R}^n$ is defined as follows. For a rectangular parallelepiped $Q = [a_1,b_1] \times \cdots \times [a_n,b_n]$ we set $\operatorname{Vol}(Q) = \prod_{j=1}^{n}(b_j - a_j)$. Then
$$\mathcal{L}^n(A) := \inf \sum_{i=1}^{\infty} \operatorname{Vol}(Q_i),$$
where the infimum is taken over all countable collections of rectangular parallelepipeds $\{Q_i : i = 1,2,\ldots\}$ such that $A \subset \bigcup_{i=1}^{\infty} Q_i$. (If $A$ is the empty set we declare $\mathcal{L}^n(A) = 0$.) Lebesgue measure is a $\sigma$-finite doubling Radon measure. Moreover, $\mathcal{L}^n(A) = \operatorname{Vol}(Q)$ for each parallelepiped $Q$. To see that Lebesgue measure is doubling, it suffices to observe that
$$\mathcal{L}^n(B(x,r)) = \mathcal{L}^n(\overline{B(x,r)}) = \Omega_n r^n$$ 
for all balls $B(x,r)$ in $\mathbb{R}^n$, where $\Omega_n$ denotes a constant depending only on the dimension $n$. (In fact, $\Omega_n$ is the volume of the unit ball of $\mathbb{R}^n$.)
Example 1.2.10 (Restrictions of measures). Let \( \mu \) be a measure on a set \( X \) and let \( Y \) be a fixed subset of \( X \). The restriction of \( \mu \) to \( Y \), written \( \mu|_Y \), is the measure defined by 
\[
(\mu|_Y)(A) = \mu(Y \cap A), \quad A \subset X.
\]

Example 1.2.11 (Pushforward measures). Let \( f : X \to Y \) be a function and let \( \mu \) be a measure on \( X \). The pushforward of \( \mu \) by \( f \), written \( f_\# \mu \), is the measure on \( Y \) defined by 
\[
f_\# \mu(A) = \mu(f^{-1}(A)) \quad \text{for } A \subset Y.
\] If \( \mu \) is a Borel measure and \( f \) is a Borel function, then \( f_\# \mu \) is a Borel measure. See [GMT, pp. 16–17] for a proof of the following facts:

• If \( \mu \) is a Radon measure on \( X \) with compact support and \( f : X \to Y \) is continuous, then \( f_\# \mu \) is a Radon measure on \( Y \).

• Let \( X \) and \( Y \) be compact metric spaces and let \( f : X \to Y \) be a continuous surjection. For any Radon measure \( \nu \) on \( Y \) there exists a Radon measure \( \mu \) on \( X \) so that \( f_\# \mu = \nu \).

Example 1.2.12. Let \( V \) be an \( m \)-dimensional subspace of \( \mathbb{R}^n \). Then \( V \) can be identified with \( \mathbb{R}^m \) by fixing a basis of \( V \). More precisely, there exists a linear bijection \( f : \mathbb{R}^m \to V \). Via \( f \) we may push the Lebesgue measure from \( \mathbb{R}^k \) to a measure on \( V \). Thus we consider the measure \( f_\# \mathcal{L}^m \) on \( V \). We will continue to call this the Lebesgue \( m \)-measure and will continue to denote it by \( \mathcal{L}^m \).

Example 1.2.13. Let \( G \) be a metric group (i.e., a group equipped with a metric \( d : G \times G \to [0, \infty) \) such that the map \((x, y) \to xy^{-1}\) is continuous). A measure \( \mu \) on \( G \) is left invariant if \( \mu(gA) = \mu(A) \) for all \( A \subset G \), where \( gA = \{gx : x \in A\} \). A measure \( \mu \) is called a left Haar measure if it is a left invariant Radon measure. Right invariant measures and Haar measures are defined similarly. A measure is invariant if it is both left and right invariant, and is a Haar measure if it is both a left and a right Haar measure.

Every locally compact metric group supports a left Haar measure. Moreover, any two left Haar measures are proportional, i.e., if \( \mu \) and \( \nu \) are left Haar measures on a locally compact metric group \( G \), then \( \mu = c\nu \) for some constant \( c > 0 \). If the group \( G \) is compact, it supports a finite Haar measure \( \mu \) which is again unique up to constant multiples. In the latter case, it is traditional to require that \( \mu \) be a probability measure, i.e., \( \mu(G) = 1 \).

Haar measure can constructed in arbitrary (locally) compact topological groups. There are some subtleties in the theory of Haar measures on topological groups which are not present in the metric setting. All of the examples we will consider in this course will in fact be metric groups.

A property \( P \) of elements of \( X \) is said to hold \( \mu \)-almost everywhere (\( \mu \)-a.e.) if the set \( S \) of points \( x \in X \) at which property \( P \) does not hold satisfies \( \mu(S) = 0 \). For instance, \( \mathcal{L}^1 \)-a.e. real number is irrational.

1.3. Integration in measure spaces. Integrals\(^1\)

\[
\int_A f \, d\mu = \int_A f(x) \, d\mu x
\]

of a \( \mu \)-measurable function \( f : X \to [-\infty, \infty] \) over a measurable set \( A \) with respect to a measure \( \mu \) are defined in the standard way. Recall that such integrals are defined first

\(^1\) We adopt a notational convention from [M], writing the element of integration \( d\mu x \) without any parentheses.
for non-negative (extended real-valued) functions $f : X \to [0, \infty]$ and then extended to all functions by the rule

$$\int_A f \, d\mu := \int_A f^+ \, d\mu - \int_A f^- \, d\mu,$$

where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$; note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. The value in (1.3.1) is well-defined only in case either $\int_A f^+ \, d\mu$ or $\int_A f^- \, d\mu$ is finite. The space $L^1(\mu)$ of $\mu$-integrable functions consists of all $\mu$-measurable functions $f$ such that $\int_A |f| \, d\mu < \infty$. When $A = X$ we often omit the subscript on the integral sign.

We abbreviate $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f(x) \, dx$ for integrals with respect to Lebesgue measure $\mathcal{L}^n$.

For $1 \leq p < \infty$ we introduce the $L^p$ space $L^p(\mu)$ to be the space of $\mu$-measurable functions $f$ such that $\int |f|^p \, d\mu < \infty$. Also $L^\infty(\mu)$ denotes the space of $\mu$-essentially bounded functions; $f \in L^\infty(\mu)$ if there exists $M < \infty$ so that $|f| \leq M$ $\mu$-a.e.

The following result is an easy consequence of the definitions. It relates integrals with respect to a measure $\mu$ and its pushforward under a Borel map $f : X \to Y$.

**Theorem 1.3.2.** Let $f : X \to Y$ be a Borel mapping, let $\mu$ be a Borel measure on $X$, and let $g$ be a non-negative Borel function on $Y$. Then $\int_Y g \, df_{\#}\mu = \int_X (g \circ f) \, d\mu$.

We will make use of many basic properties of integrals in measure spaces such as convergence theorems, Fubini’s theorem, Hölder’s inequality, and so on. For statements and proofs of these results, please consult any standard reference on real analysis and measure theory.

### 1.4. Carathéodory’s construction and Hausdorff measures.

We present Carathéodory’s method for constructing Borel measures in metric spaces.

**Definition 1.4.1.** Let $(X, d)$ be a separable metric space, let $\mathcal{F}$ be a family of Borel subsets of $X$, and let

$$h : [0, \infty) \to [0, \infty)$$

be a non-decreasing continuous function with $h(0) = 0$. For $\delta > 0$ and $A \subset X$, let

$$\mathcal{H}_\delta^h(A) = \inf \sum_i h(\text{diam } E_i),$$

where the infimum is taken over all coverings $E_1, E_2, \ldots$ of $A$ by elements from the class $\mathcal{F}$ with diameter at most $\delta$. (When $A = \emptyset$ we declare $\mathcal{H}_\delta^h(\emptyset) = 0$.)

It is clear that $\mathcal{H}_\delta^h(A)$ increases as $\delta \searrow 0$ (for fixed $h$ and $A$). Set

$$\mathcal{H}^h(A) = \lim_{\delta \searrow 0} \mathcal{H}_\delta^h(A) = \sup_{\delta > 0} \mathcal{H}_\delta^h(A).$$

In order for the preceding definition of $\mathcal{H}_\delta^h$ to be reasonable, we need to impose the following assumption on the family $\mathcal{F}$: for each $\delta > 0$, $X$ can be covered by countably many elements of $\mathcal{F}$ each with diameter less than $\delta$. This assumption is typically easy to verify. Standard families $\mathcal{F}$ which we consider include the full power set $\mathcal{P}(X)$, the family of all balls $B(x, r)$, $x \in X$, $r > 0$, or (in the case $X = \mathbb{R}^n$) the family of all rectangular parallelepipeds.

It is easy to check that $\mathcal{H}_\delta^h$ is monotonic and countably subadditive, hence a measure on $X$. In rough terms, it measures the “size” of $A$ from the perspective of efficient coverings of $A$ by sets of size at most $\delta$, relative to the cost function $\sum_i h(\text{diam } E_i)$.

When $\delta > 0$, $\mathcal{H}_\delta^h$ is typically not additive on disjoint Borel sets. However, $\mathcal{H}^h$ is a Borel measure. This follows from the next proposition.
Proposition 1.4.2. For any metric space \((X,d)\), any family \(\mathcal{F}\), and any suitable function \(h\), the measure \(\mathcal{H}^h\) satisfies Carathéodory’s criterion (1.2.4). Moreover, \(\mathcal{H}^h\) is Borel regular.

Proof. It suffices to prove that \(\mathcal{H}^h(A \cup B) \geq \mathcal{H}^h(A) + \mathcal{H}^h(B)\) whenever \(A, B \subset X\) satisfy \(\text{dist}(A,B) > 0\). Fix \(0 < \delta < \frac{1}{3} \text{dist}(A,B)\). Suppose that \(A \cup B\) is covered by a countable collection of sets \(E_1, E_2, \ldots\) from \(\mathcal{F}\), with \(\text{diam} E_i < \delta\) for all \(i\). Since \(\text{diam} E_i < \delta\) and \(\text{dist}(A,B) > 3\delta\), none of the sets \(E_i\) can intersect both \(A\) and \(B\). Hence

\[
\sum_{i=1}^{\infty} h(\text{diam} E_i) \geq \sum_{E_i \cap A \neq \emptyset} h(\text{diam} E_i) + \sum_{E_i \cap B \neq \emptyset} h(\text{diam} E_i) \geq H^h_\delta(A) + H^h_\delta(B).
\]

Taking the infimum over all such coverings of \(A \cup B\) gives \(H^h_\delta(A \cup B) \geq H^h_\delta(A) + H^h_\delta(B)\) and letting \(\delta \to 0\) gives the desired inequality.

The fact that \(\mathcal{H}^h\) is Borel regular comes from the construction of \(\mathcal{H}^h\) and the fact that the elements of \(\mathcal{F}\) are Borel sets.

Example 1.4.3. Let \(h_s(t) = t^s\) for some \(0 \leq s < \infty\), and let \(\mathcal{F}\) be the class of all Borel subsets of \(X\). The measure \(\mathcal{H}^{h_s}\) is called the \(s\)-dimensional Hausdorff measure on \(X\). In this setting we write \(\mathcal{H}^s\) in place of \(\mathcal{H}^{h_s}\).

Exercise 1.4.4. Show that \(\mathcal{H}^0\) coincides with the counting measure on any metric space.

Exercise 1.4.5. Show that \(\mathcal{H}^1 = \mathcal{L}^1\) on the real line \(\mathbb{R}\).

Exercise 1.4.6. For \(A \subset \mathbb{R}^n, v \in \mathbb{R}^n, r > 0\) and \(s \geq 0\), show that \(\mathcal{H}^s(A + v) = \mathcal{H}^s(A)\) and \(\mathcal{H}^s(ra) = r^s \mathcal{H}^s(A)\). Here \(A + v = \{a + v : a \in A\}\) and \(ra = \{ra : a \in A\}\).

The following proposition is fundamental. Computing the exact proportionality constant is surprisingly difficult, relying on the isodiametric inequality for \(\mathbb{R}^n\). Some sources define the Hausdorff measures by including an additional multiplicative constant designed to make the measures \(\mathcal{H}^n\) and \(\mathcal{L}^n\) equal in each dimension \(n\). We do not adopt this approach.

Proposition 1.4.7. For each \(n\), there exists \(c(n) > 0\) so that \(\mathcal{H}^n = c(n) \mathcal{L}^n\) in \(\mathbb{R}^n\).

Proof. Both \(\mathcal{H}^n\) and \(\mathcal{L}^n\) are Haar measures on \(\mathbb{R}^n\) (viewed as an abelian group under vector addition). See Exercise 1.4.6. The conclusion follows from the uniqueness of Haar measures up to multiplicative constants.

The class of test sets \(\mathcal{F}\) can be varied. For example, choosing \(\mathcal{F}\) to be the set of all balls in \(X\) and \(h = h_s\) gives rise to the \(s\)-dimensional spherical Hausdorff measure \(\mathcal{S}^s\). It is not difficult to see that

\[
\mathcal{H}^s(A) \leq \mathcal{S}^s(A) \leq 2^s \mathcal{H}^s(A)
\]

for any \(A \subset X\) and any \(0 \leq s < \infty\). Indeed, the left hand inequality follows immediately, since the infimum in the definition of \(\mathcal{S}^s\) is taken over a restricted class of sets. To prove the right hand inequality, let \(\delta > 0\) and let \(A \subset \bigcup_i E_i\) for some collection of Borel sets \(E_1, E_2, \ldots\) with \(\text{diam} E_i < \delta\). For each \(i\) choose any point \(x_i \in E_i\). Then \(E_i \subset B_i := B(x_i, \text{diam} E_i)\) and \(\text{diam} B_i \leq 2 \text{diam} E_i < 2\delta\). It follows that

\[
\mathcal{S}^s_{2\delta}(A) \leq \sum_i (\text{diam} B_i)^s \leq 2^s \sum_i (\text{diam} E_i)^s
\]

and so \(\mathcal{S}^s_{2\delta}(A) \leq 2^s \mathcal{H}^s_\delta(A) \leq 2^s \mathcal{H}^s(A)\). Taking the limit as \(\delta \to 0\) finishes the proof.
Another important variant in the Euclidean setting is the family of \textit{dyadic Hausdorff measures}. First, we introduce the dyadic decomposition of $\mathbb{R}^n$. For each integer $m$, we let $Q_m$ denote the family of half-open cubes

$$Q = \left[ \frac{k_1}{2^m}, \frac{k_1 + 1}{2^m} \right] \times \cdots \times \left[ \frac{k_n}{2^m}, \frac{k_n + 1}{2^m} \right],$$

where $k_1, \ldots, k_n \in \mathbb{Z}$. For fixed $m$, the family $Q_m$ of all such cubes defines a partition of $\mathbb{R}^n$ into disjoint sets. We call elements of $Q_m$ \textit{dyadic cubes of scale} $2^{-m}$. Note that

$$\text{diam } Q = \frac{\sqrt{n}}{2^m} \quad \forall Q \in Q_m.$$

For $s \geq 0$, the \textit{dyadic} $s$-dimensional Hausdorff measure in $\mathbb{R}^n$, denoted $\mathcal{H}^s_{\text{dyadic}}$, is the measure obtained by the preceding construction of Carathéodory for $h = h_s$, with $\mathcal{F}$ equal to the family of all dyadic cubes (of all scales).

Coverings with dyadic cubes have an important advantage over coverings with balls or more general sets. If a set $A \subset \mathbb{R}^n$ is covered by a collection $Q_1, Q_2, \ldots$ of dyadic cubes (not necessarily all of the same scale), then $A$ is covered by a subcollection $Q_{i_1}, Q_{i_2}, \ldots$ which is pairwise disjoint. This fact can be deduced from the following property of dyadic cubes: \textit{any two dyadic cubes in} $\mathbb{R}^n$ \textit{are either pairwise disjoint, or one is contained in the other}.

As was the case for spherical Hausdorff measures, dyadic Hausdorff measures on $\mathbb{R}^n$ are comparable to the corresponding Hausdorff measures (defined by arbitrary coverings).

**Proposition 1.4.9.** For any $A \subset \mathbb{R}^n$ and $s \geq 0$,

$$\mathcal{H}^s(A) \leq \mathcal{H}^s_{\text{dyadic}}(A) \leq 3^n 2^s n^{s/2} \mathcal{H}^s(A).$$

**Proof.** The left hand inequality is trivial. For the right hand inequality, fix a set $A \subset \mathbb{R}^n$, fix $\delta > 0$, and consider a covering \{$E_i : i = 1, 2, \ldots$\} of $A$ by arbitrary sets with $\text{diam } E_i < \delta$ for all $i$. For each $i$, choose an integer $m_i$ so that

$$\frac{1}{2m_i + 1} \leq \text{diam } E_i < \frac{1}{2m_i}.$$ 

We claim that $E_i$ is covered by $3^n$ dyadic cubes of scale $2^{-m_i}$. To see this, note that if some dyadic cube $Q$ of scale $2^{-m_i}$ intersects $E_i$, then $E_i$ is contained in the union of all dyadic cubes $Q'$ of scale $2^{-m_i}$ such that $Q' \cap \bar{Q} \neq \emptyset$, and the number of such cubes is $3^n$. For the latter claim, it suffices to note that if $x \in E_i \cap Q$ and $y \in E_i$ is not in any such cube $Q'$, then

$$\frac{1}{2m_i} \leq |x - y| \leq \text{diam } E_i$$

which contradicts the choice of $m_i$. To each $E_i$ we may therefore associate $3^n$ dyadic cubes $Q'$ of scale $2^{-m_i}$ whose union contains $E_i$. The diameter of any of these cubes satisfies

$$\text{diam } Q' = \frac{\sqrt{n}}{2m_i} \leq 2\sqrt{n} \text{ diam } E_i \leq 2\sqrt{n} \delta.$$

Let $\epsilon := 2\sqrt{n} \delta$. Then

$$\mathcal{H}^s_{\text{dyadic}, \epsilon}(A) \leq \sum_i \sum_{Q' \text{ assoc to } E_i} (\text{diam } Q')^s \leq 3^n (2\sqrt{n})^s \sum_i (\text{diam } E_i)^s$$

and so $\mathcal{H}^s_{\text{dyadic}, \epsilon}(A) \leq 3^n 2^s n^{s/2} \mathcal{H}^s_\delta(A) \leq 3^n 2^s n^{s/2} \mathcal{H}^s(A)$. Letting $\delta$ (and hence also $\epsilon$) tend to zero yields the right hand inequality and completes the proof of the proposition. \qed
1.5. Hausdorff dimension and behavior of Hausdorff measure/dimension under mappings. The following lemma describes how the Hausdorff measure changes as the exponent $s$ varies. It is critical for the definition of the Hausdorff dimension.

**Lemma 1.5.1.** Let $A \subset X$ and let $0 \leq s < t < \infty$. Then

(i) $\mathcal{H}^s(A) < \infty$ implies $\mathcal{H}^t(A) = 0$, and

(ii) $\mathcal{H}^t(A) > 0$ implies $\mathcal{H}^s(A) = \infty$.

Of course, (i) and (ii) are equivalent. Taken together, they show that the value $\mathcal{H}^s(A)$ can be positive and finite for at most one value $s$. This value is the Hausdorff dimension of the set $A$.

**Definition 1.5.2.** The Hausdorff dimension of $A \subset X$ is the unique value $s_0 \in [0, \infty)$ so that $\mathcal{H}^s(A) = 0$ for all $s > s_0$ and $\mathcal{H}^s(A) = \infty$ for all $0 \leq s < s_0$. We will write $H \dim(A)$ for the Hausdorff dimension of $A$. If necessary we write $H \dim_d(A)$ to indicate the dependence of this value on the metric $d$.

Here are some simple properties of this dimension:

- **monotonicity:** $H \dim(A) \leq H \dim(B)$ if $A \subset B$;
- **countable stability:** $H \dim(A) = \sup_i H \dim(A_i)$ if $A_1, A_2, \cdots \subset X$ and $A = \bigcup_i A_i$;
- $H \dim(A) = 0$ for every countable set $A$;
- $H \dim(A) = m$ whenever $A$ is a smooth $m$-dimensional submanifold of $\mathbb{R}^n$.

Notice that the value of the Hausdorff dimension is unchanged if we replace the Hausdorff measures with the spherical Hausdorff measures. This follows from (1.4.8).

**Definition 1.5.3.** A map $f : X \to Y$ between metric spaces $(X, d_X)$ and $(Y, d_Y)$ is said to be $L$-Lipschitz if

$$d_Y(f(x), f(y)) \leq Ld_X(x, y)$$

for every $x, y \in X$. A homeomorphism $f : X \to Y$ is said to be $L$-bi-Lipschitz if

$$\frac{1}{L}d_X(x, y) \leq d_Y(f(x), f(y)) \leq Ld_X(x, y)$$

for every $x, y \in X$. A map $f : X \to Y$ is said to be an $L$-bi-Lipschitz embedding if it is an $L$-bi-Lipschitz homeomorphism from $X$ onto its image $f(X) \subset Y$.

The behavior of Hausdorff measures and dimensions under Lipschitz maps is easy to observe.

**Proposition 1.5.4.** Let $f : (X, d) \to (Y, d')$ be $L$-Lipschitz and let $A \subset X$. Then

(i) $\mathcal{H}^s(f(A)) \leq L^s\mathcal{H}^s(A)$ for every $0 \leq s < \infty$;

(ii) $H \dim f(A) \leq H \dim A$.

**Corollary 1.5.5.** Let $f : (X, d) \to (Y, d')$ be bi-Lipschitz. Then $H \dim f(A) = H \dim A$ for every $A \subset X$.

Thus Hausdorff dimension is a **bi-Lipschitz invariant** of metric spaces. One of the central problems in analysis in metric spaces is to determine a list of metric properties or quantities which suffice to characterize spaces up to bi-Lipschitz equivalence (or up to equivalence by mappings in some other natural metrically defined class).

**Exercise 1.5.6.** Show that if $f : (X, d) \to (Y, d')$ satisfies $d'(f(x_1), f(x_2)) = Ld(x_1x_2)$ for all $x_1, x_2 \in X$, then $\mathcal{H}(f(A)) = L^s\mathcal{H}(A)$ for all $A \subset X$ and all $s \geq 0$. 

1.6. An example: the Cantor set. To illustrate the computation of Hausdorff measure and dimension, we consider the classical Cantor set

\[ C = [0, 1] \setminus \left( \left( \frac{1}{3}, \frac{2}{3} \right) \cup \left( \frac{1}{9}, \frac{2}{9} \right) \cup \left( \frac{7}{9}, \frac{8}{9} \right) \cup \cdots \right). \]

We may write \( C = C_L \cup C_R \) where \( C_L \) and \( C_R \) are similarity copies of \( C \), scaled by ratio \( \frac{1}{3} \). More precisely, \( C_L = \frac{1}{3}C \) and \( C_R = \frac{1}{3}C + \frac{2}{3} \) (where the notation is as in Exercise 1.4.6). If \( s_0 \) denotes the Hausdorff dimension of \( C \) and we assume that \( 0 < H^{s_0}(C) < \infty \), then

\[ H^{s_0}(C) = H^{s_0}(C_L) + H^{s_0}(C_R) = 2\left( \frac{1}{3} \right)^{s_0} H^{s_0}(C) \]

and we deduce that \( 2\left( \frac{1}{3} \right)^{s_0} = 1 \) and

\[(1.6.1) \quad H \dim(C) = s_0 := \frac{\log 2}{\log 3}.\]

In order to make this computation rigorous we must justify the italicized assumption. Here are the details.

Consider finite words \( w \) in the letters \( L \) and \( R \), i.e., \( w = w_1w_2 \cdots w_m \) with \( w_i \in \{ L, R \} \). We call \( m \) the length of such a word, and write \( W_m \) for the set of all words of length \( m \). Each word \( w \in W_m \) generates a subset \( C_w \subset C \) by the obvious iterative procedure, and

\[ C = \bigcup_{w \in W_m} C_w \]

for each \( m \). Note that \( \text{diam} C_w = 3^{-m} \) for each \( w \in W_m \). It follows that

\[ H^{s_0}_{3^{-m}}(C) \leq \sum_{w \in W_m} (\text{diam} C_w)^{s_0} = 2^m \cdot 3^{-ms_0} = 1. \]

Letting \( m \to \infty \) gives \( H^{s_0}(C) \leq 1 \) so \( H \dim C \leq s_0 \).

On the other hand, suppose that \( \{ E_i \} \) is an arbitrary cover of \( C \). By replacing \( E_i \) with its convex hull and expanding slightly if necessary, we may assume that the \( E_i \)'s are open intervals; since \( C \) is compact we may assume that there are only finitely many \( E_i \)'s.

For each \( i \), let \( m_i \) be the integer such that

\[(1.6.2) \quad 3^{-m_i - 1} \leq \text{diam} E_i < 3^{-m_i}. \]

For distinct words \( w, w' \) in \( W_m \), we have \( \text{dist}(C_w, C_{w'}) \geq 3^{-m} \). It follows that each interval \( E_i \) can meet at most one set \( C_w, w \in W_{m_i}, \) and so can meet at most \( 2^{m-m_i} \) such sets \( C_w, w \in W_m, m \geq m_i. \) Since all \( 2^m \) sets \( C_w, w \in W_m \) are met, we have

\[ 2^m \leq \sum_i 2^{m-m_i} \leq 3^{s_0} \cdot 2^m \sum_i (\text{diam} E_i)^{s_0}, \]

where the second inequality comes from (1.6.2). Hence

\[ H^{s_0}(C) \geq H^{s_0}_{\infty}(C) \geq 3^{-s_0} = \frac{1}{2} \]

and so \( H \dim C \geq s_0 \). Now (1.6.1) is justified.

Remark 1.6.3. Even in this simple example, the computation of lower bounds for Hausdorff measures was somewhat involved. In the next few lectures, we will discuss a variety of methods for obtaining lower bounds for Hausdorff measures.
1.7. Hausdorff dimension vs. topological dimension. The small inductive dimension \(\dim X\) of a topological space \(X\) is defined as follows: the empty set is declared to have dimension \(-1\), and a topological space \(X\) has dimension \(\leq n\) if each point in \(X\) has arbitrarily small neighborhoods whose boundaries have dimension \(\leq n - 1\). For example, \(\mathbb{R}^n\) has small inductive dimension \(n\), while the Cantor set \(C\) has dimension zero.

Remark 1.7.1. The preceding is only one of several definitions of dimension in topological spaces. Other possibilities include the (Lebesgue) covering dimension and the large inductive dimension. For separable metric spaces, all of these definitions are equivalent. We will always work in this setting; we call \(\dim X\) the topological dimension of \(X\).

Observe that in contrast with Hausdorff dimension, the topological dimension is a purely topological notion; homeomorphically equivalent metric spaces have equal topological dimension. The relation between these two notions is contained in the following theorem.

Theorem 1.7.2. Let \((X, d)\) be a metric space. Then \(H \dim X \geq \dim X\).

More precisely, if \(X\) is a metrizable topological space, then \(\dim X \leq H \dim X\) for any metric \(d\) on \(X\) which generates the topology. It can be shown that

\[
(1.7.3) \quad \dim X = \inf_d H \dim_d X
\]

where the infimum is taken over all metrics \(d\) on \(X\) which generate the topology. We will not prove or use (1.7.3) in this course.

Before proving Theorem 1.7.2, let us consider a special case.

Proposition 1.7.4. Let \((X, d)\) be a metric space with \(H \dim X < 1\). Then \(\dim X = 0\).

Proof of Proposition 1.7.4. Let \(x_0\) be a fixed point in \(X\) and consider the 1-Lipschitz function \(f\) on \(X\) given by \(f(x) = d(x, x_0)\). By Proposition 1.5.4, \(H \dim f(X) < 1\). Thus

\[
\mathcal{L}^1(f(X)) = \mathcal{H}^1(f(X)) = 0;
\]

recall that \(\mathcal{L}^1\) denotes the Lebesgue measure in \(\mathbb{R}\). It follows that \(f(X)\) contains no open neighborhood of \(0 = f(x_0)\), i.e., there exists \(r_n \to 0, r_n \notin f(X)\). Then the “metric spheres”

\[
S(x_0, r) := \{y \in X : d(x_0, y) = r_n\}, \quad n = 1, 2, \ldots,
\]

are all empty. Since the boundary of \(B(x_0, r)\) is contained in \(S(x_0, r)\), we conclude that \(x_0\) has arbitrarily small neighborhoods with empty boundary. Since \(x_0\) was arbitrary the conclusion follows.

To prove Theorem 1.7.2, it suffices to show the following lemma. Lemma 1.7.5 can be viewed as a “Fubini-type” theorem for Hausdorff measures in metric spaces.

Lemma 1.7.5. Suppose that \(\mathcal{H}^{1+s}(X) = 0\) for some \(s \in \mathbb{Z}, s \geq -1\). Then (i) \(\mathcal{H}^s(\partial B(x_0, r)) = 0\) for all \(x_0 \in X\) and a.e. \(r > 0\), and (ii) \(\dim X \leq s\).

Proof of Lemma 1.7.5. Assume first that (i) has been proved. We prove (ii) by induction on \(s\). The case \(s = -1\) is obvious. Suppose that the lemma holds for some integer \(s - 1\) and all metric spaces, and let \((X, d)\) be a metric space such that \(\mathcal{H}^{1+s}(X) = 0\). By (i), \(\mathcal{H}^s(\partial B(x_0, r)) = 0\) for all \(x_0 \in X\) and a.e. \(r > 0\). By the inductive hypothesis, \(\dim \partial B(x_0, r) \leq s - 1\) for these \(x_0\) and \(r\). Thus every \(x_0 \in X\) has a neighborhood basis consisting of sets \(B(x_0, r)\) whose boundary has dimension \(\leq s - 1\), so \(\dim X \leq s\).
It remains to show that (i) holds. Fix $x_0 \in X$ and a ball $B \subset X$. Define $r_{\pm}$ by

$$r_- = \inf_{x \in B} d(x_0, x) = \text{dist}(x_0, B) \quad \text{and} \quad r_+ = \sup_{x \in B} d(x_0, x).$$

Note that $r_+ - r_- \leq \text{diam} B$ by the triangle inequality. Since $\partial B(x_0, r)$ meets $B$ only for $r \in [r_-, r_+]$, we have

$$\int_0^\infty (\text{diam } \partial B(x_0, r) \cap B)^s \, dr = \int_{r_-}^{r_+} (\text{diam } \partial B(x_0, r) \cap B)^s \, dr \leq (\text{diam } B)^s (r_+ - r_-) \leq (\text{diam } B)^{1+s}.$$

Since $\mathcal{H}^{1+s}(X) = 0$ we can choose a sequence of balls $B_1, B_2, \ldots$ with $\text{diam } B_i < \epsilon$ and

$$\sum_i (\text{diam } B_i)^{1+s} < \epsilon.$$

Hence

$$\epsilon > \sum_i \int_0^\infty (\text{diam } \partial B(x_0, r) \cap B_i)^s \, dr \geq \int_0^\infty \sum_{B_i \cap \partial B(x_0, r) \neq \emptyset} (\text{diam } \partial B(x_0, r) \cap B_i)^s \, dr \geq \int_0^\infty \mathcal{H}^s(\partial B(x_0, r)) \, dr.$$

Letting $\epsilon \to 0$ gives $\int_0^\infty \mathcal{H}^s(\partial B(x_0, r)) \, dr = 0$ so $\mathcal{H}^s(\partial B(x_0, r)) = 0$ for a.e. $r > 0$. \hfill \Box

**Exercise 1.7.6.** Let $\mathcal{H}$ be the family of all non-decreasing functions $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$. Define a partial ordering $h \prec k$ on $\mathcal{H}$ by the requirement $h(t)/k(t) \to 0$ as $t \to 0$. Show that $h \prec k$ and $\mathcal{H}^k(X) < \infty \Rightarrow \mathcal{H}^h(X) = 0$. Conclude that the generalized Hausdorff measure $\mathcal{H}^h(X)$ can be positive and finite for at most one such function $h$ within any totally ordered subclass of $\mathcal{H}$.

**Exercise 1.7.7.** For any $s > 0$ and any $(X, d)$, show that $\mathcal{H}^s(X) = 0$ iff $\mathcal{H}^s_\infty(X) = 0$.

**Exercise 1.7.8.** For any connected metric space $(X, d)$, show that $\text{diam } X \leq \mathcal{H}^{1}_\infty(X)$.

### 1.8. References for further reading

Our primary reference in this course is


Some of the material in this section can be found in Chapters 1 and 4 of [M].

Other good references for Hausdorff measures and their applications include


The classical reference for topological dimension theory is


Some of the material in subsection 1.7 is taken from [HW].