Long-term behavior of solutions
to first-order linear ODE's with constant coefficients

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The general first-order linear ordinary differential equation is
\[ y' + P(x)y = Q(x), \quad y(0) = y_0, \quad (1) \]
where \( P(x) \) and \( Q(x) \) are given functions of \( x \). As we discussed in class, this initial value problem has a unique solution, defined over any interval on which both \( P(x) \) and \( Q(x) \) are continuous, which is given by the formula
\[ y(x) = e^{-\int P(u)\,du} \left( y_0 + \int_0^x Q(u)e^{\int P(v)\,dv\,du} \right). \quad (2) \]

In this handout, we consider the special case of (1) when the function \( P \) is constant. We also write \( t \) for the independent variable instead of \( x \), since in many applications it corresponds to a time variable. The ODE
\[ y' + ry = f(t), \quad y(0) = y_0, \quad (3) \]
where \( r \) is a constant, is called a constant-coefficient differential equation. The function \( f(t) \) which occurs on the right hand side is called the forcing function; typically it corresponds to an external force which is exerted on a physical system.

In this setting the integrating factor is \( e^{\int P(u)\,du} = e^{rt} \) and the solution in (2) takes the simpler form
\[
y(t) = e^{-rt} \left( y_0 + \int_0^t Q(u)e^{ru\,du} \right) = y_0e^{-rt} + e^{-rt} \int_0^t Q(u)e^{ru\,du}.
\quad (4)
\]

It is useful to think of this expression as the combination of two terms:
\[ y(t) = y_{\text{unf}}(t) + y_{\text{zi}}(t), \]
where the unforced term \( y_{\text{unf}}(t) = y_0e^{-rt} \) solves the equation
\[ y'_{\text{unf}} + ry_{\text{unf}} = 0, \quad y_{\text{unf}}(0) = y_0, \quad (5) \]
and the zero-input term \( y_{\text{zi}}(t) = e^{-rt} \int_0^t Q(u)e^{ru\,du} \) solves the equation
\[ y'_{\text{zi}} + ry_{\text{zi}} = f(t), \quad y_{\text{zi}}(0) = 0. \quad (6) \]
Notice that in (5) the forcing function \( f(t) \) has been removed; this explains the name “unforced” for this term. Also in (6) the initial value \( y(0) \) has been set to zero; this explains the name “zero-input” for this term.

When \( r > 0 \), \( y_{\text{unf}}(t) = y_0e^{-rt} \) is a decaying exponential, and tends to zero as \( t \to \infty \). Therefore, all solutions to (3) approach the graph of the zero-input solution \( y_{\text{unf}}(t) \) when \( t \) becomes very large. In this case, the unforced term represents a contribution to the behavior of the system which dies out rapidly as time progresses, while the zero-input term represents the long-term or eventual behavior of the system. For this reason we sometimes call \( y_{\text{unf}}(t) \) the names transient solution and \( y_{\text{zi}}(t) \) the steady-state (or equilibrium) solution. (Warning! This terminology only makes sense if \( r > 0 \). When \( r < 0 \), \( y_{\text{unf}}(t) = y_0e^{-rt} \) is an increasing exponential, and solutions to (3) of the form (4) grow farther and farther apart as \( t \) tends to \( \infty \).)

The principle of superposition for (first-order) linear equations states the following: if \( y_1(t) \) is a solution to
\[ y'_1 + ry_1 = f(t), \quad y_1(0) = a, \]
and \( y_2(t) \) is a solution to
\[ y'_2 + ry_2 = g(t), \quad y_2(0) = b, \]
then \( y_3(t) = y_1(t) + y_2(t) \) is a solution to
\[ y'_3 + ry_3 = f(t) + g(t), \quad y_3(0) = a + b. \]

In practice, what this means is that we can add together solutions to certain (simple) linear equations to generate solutions to more complicated equations. Put another way, we can split a complicated equation into simpler pieces, solve each piece separately, and combine the resulting solutions to get the solution to the original problem. The decomposition of the solution \( y(t) \) in (4) for the equation in (3) into the unforced and zero-input terms is an example of this idea.

The concepts of linearity and superposition will be very important later on in the course, when we study higher-order linear ODE’s and linear PDE’s.