Homogeneous Linear Equations of Higher Order — Summary

The general homogeneous linear differential equation takes the form
\[ a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0. \] (§)

We always assume that the coefficient functions \( a_0(x), \ldots, a_n(x) \) are continuous functions defined on some interval.

Solutions to (§) satisfy the important principle of superposition, which states that linear combinations of solutions are again solutions:

If \( y_1, \ldots, y_n \) solve (§) and \( c_1, \ldots, c_n \) are constants, then \( c_1y_1 + \cdots + c_n y_n \) also solves (§).

**Existence and Uniqueness Theorem:** If \( a_0(x), \ldots, a_{n-1}(x) \) are continuous functions on an interval containing a point \( x_0 \) and \( b_0, \ldots, b_{n-1} \) are constants, then the initial value problem
\[ y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0, \]
\[ y(x_0) = b_0, y'(x_0) = b_1, \ldots, y^{(n-1)}(x_0) = b_{n-1}, \]
has a unique solution on that interval.

A collection of functions \( f_1, \ldots, f_n \) is linearly dependent on an interval if there exist constants \( c_1, \ldots, c_n, \) **not all equal to zero**, so that \( c_1 f_1(x) + \cdots + c_n f_n(x) = 0 \) for every \( x \) in the interval. If \( f_1, \ldots, f_n \) are not linearly dependent, we say that they are linearly independent.

An easy way to test whether a collection of functions is linearly independent is to use the Wronskian. The Wronskian of \( n \) functions \( f_1, \ldots, f_n \) is the new function
\[
W(f_1, \ldots, f_n) = \begin{vmatrix}
  f_1 & f_2 & \cdots & f_n \\
  f_1' & f_2' & \cdots & f_n' \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)}
\end{vmatrix}.
\]
(This symbol represents the determinant of the matrix with these entries; for a quick review of matrices and determinants see the next page of this handout.)

**Theorem:** Let \( y_1, \ldots, y_n \) be solutions to a homogeneous linear differential equation
\[ y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0 \] (†)
on an interval where the coefficient functions \( a_0(x), \ldots, a_{n-1}(x) \) are continuous. Then \( W(x) \) is either identically equal to zero or never equal to zero throughout the interval and:

(i) if \( W(y_1, \ldots, y_n) \) is never equal to zero, then \( y_1, \ldots, y_n \) are linearly independent. In this case every solution \( Y \) to (†) can be written as a linear combination of the functions \( y_1, \ldots, y_n \):
\[ Y = c_1 y_1 + \cdots + c_n y_n \]
for some constants \( c_1, \ldots, c_n \);

(ii) if \( W(y_1, \ldots, y_n) \) is identically equal to zero, then \( y_1, \ldots, y_n \) are linearly dependent.
Constant Coefficient Equations

A homogeneous constant coefficient equation takes the form

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \]  

(\dagger)

for some real constants \(a_0, \ldots, a_n\).

To find the solutions of such an equation, we use the characteristic equation

\[ a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0. \]  

(\dagger\dagger)

The Fundamental Theorem of Algebra tells us that this equation always has \(n\) roots provided we include complex roots and count roots according to multiplicity.

1. Every real root \(r\) of (\dagger\dagger) corresponds to a solution \(y(x) = e^{rx}\) to (\dagger).

2. If \(r_1 = p + qi\) and \(r_2 = p - qi\) are two roots of (\dagger\dagger) (occurring in a complex conjugate pair), then two linearly independent solutions to (\dagger) are \(y_1(x) = e^{px} \cos(qx)\) and \(y_2(x) = e^{px} \sin(qx)\).

3. If a root \(r\) of (\dagger\dagger) is repeated \(m\) times (or has multiplicity \(m\)), then \(m\) linearly independent solutions to (\dagger) are

\[ y_1(x) = e^{rx}, \quad y_2(x) = xe^{rx}, \quad \ldots, \quad y_m(x) = x^{m-1} e^{rx}. \]

Similarly, if a pair \(r_1 = p + qi\) and \(r_2 = p - qi\) or complex conjugate roots has multiplicity \(m\), then \(2m\) linearly independent solutions to (\dagger) are

\[ y_1(x) = e^{px} \cos(qx), \quad y_2(x) = e^{px} \sin(qx), \quad \vdots \]

\[ y_{2m-1}(x) = x^{m-1} e^{px} \cos(qx), \quad y_{2m}(x) = x^{m-1} e^{px} \sin(qx). \]

All functions which arise in this manner are linearly independent. It follows that we can always write down a full set of \(n\) linearly independent solutions to (\dagger).

Review: Determinants of Matrices

The determinant is a function from the collection of square matrices to the real numbers. It is defined recursively. For a \(2 \times 2\) matrix \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) the determinant is

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \]

For a \(3 \times 3\) matrix \(\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\) the determinant can be computed by expanding along the top row:

\[ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg) = aei + bfg + cdh - bdi - afh - ceg. \]

Notice the alternating pattern of signs. In general, for an \(n \times n\) matrix \(A\) whose \((i,j)\)th component is \(a_{ij}\), the determinant \(|A|\) is

\[ |A| = a_{11} |A_{11}| - a_{12} |A_{12}| + \cdots \pm a_{1n} |A_{1n}|, \]  

(\£)

where \(A_{ij}\) denotes the \((n-1) \times (n-1)\) matrix (the \((i,j)\)th minor of \(A\)) obtained by removing the \(i\)th row and \(j\)th column of \(A\). The determinants of the \(A_{ij}\) are computed recursively using (\£).

It is not necessary to expand along the top row. Any particular row or column of \(A\) can be used. The signs which should be used for the terms in (\£) come from the following pattern:

\[ \begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \end{pmatrix}. \]