Math 221 Recitation Exercises 6
Derivatives

(1) Use a calculator to plot \( y = f(x) = a^x \) as well as \( y = \frac{a^{x+h} - a^x}{h} \) for the choices \( h = 1, h = 0.5, h = 0.2 \) and \( h = 0.1 \) in each of the following cases. Use the window \(-4 \leq x \leq 4, 0 \leq y \leq 4\).
(a) \( a = 2 \)
(b) \( a = 3 \)
(c) \( a = 2.7 \)

(2) Consider the function \( f(x) = a^x \) for \( a > 0 \). Show that 
\[
\frac{f(x + h) - f(x)}{h} = c_1(h, a)f(x)
\]
for each \( h \neq 0 \), where \( c_1(h, a) \) is a value that depends only on \( h \) and \( a \). Then show that 
\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = c_2(a)f(x),
\]
where \( c_2(a) \) is a value that depends only on \( a \). Find a formula for \( c_2(a) \) in terms of \( c_1(h, a) \).

The base of the natural logarithm is \( e = \lim_{n \to \infty} (1 + \frac{1}{n})^n \approx 2.718281828 \ldots \). Problem 1 suggests the following result, which can be rigorously proven:

**Theorem 1.** If \( f(x) = e^x \), then \( f'(x) = e^x \).

(3) Find the derivatives of each of the following:

(a) \( f(x) = x^n e^x \) for an arbitrary positive integer \( n \).

(b) \( f(x) = \frac{1}{2}(e^x + e^{-x}) \) and \( g(x) = \frac{1}{2}(e^x - e^{-x}) \).

(c) \( f(x) = e^{g(x)} \). Your answer should involve \( g(x) \) and \( g'(x) \).
Remark 2. The hyperbolic cosine function is \( \cosh(x) = \frac{1}{2}(e^x + e^{-x}) \). The hyperbolic sine function is \( \sinh(x) = \frac{1}{2}(e^x - e^{-x}) \).

(4) Use a calculator to plot \( y = \sin(x) \) as well as \( y = \frac{\sin(x+h)-\sin(x)}{h} \) for the choices \( h = 1, h = 0.5, h = 0.2 \) and \( h = 0.1 \). Use the window \(-10 \leq x \leq 10, -2 \leq y \leq 2\). What does your graph suggest about the value of the derivative of \( y = \sin(x) \)?

Problem 4 suggests part (a) of the following result, which can be rigorously proven:

Theorem 3. (a) If \( f(x) = \sin(x) \), then \( f'(x) = \cos(x) \). (b) If \( g(x) = \cos(x) \), then \( g'(x) = -\sin(x) \).

Proof of (b). We use the trig identity \( g(x) = \cos(x) = \sin(\pi/2 - x) = f(\pi/2 - x) \).

By the Chain Rule,
\[
g'(x) = f'(\frac{\pi}{2} - x) \cdot (-1) = -\cos\left(\frac{\pi}{2} - x\right) = -\sin\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) = -\sin(x).
\]

(5) Find the derivatives of each of the following using only differentiation rules which have already been discussed in lecture or recitation.

(a) \( f(x) = \frac{\sin(x)}{\cos(x)} \).

(b) \( f(x) = e^{ax} \sin(bx) \) for real numbers \( a \) and \( b \).

(c) \( f(x) = \cos\left(\frac{x^2+1}{\sin(x)}\right) \).