Math 221 Recitation Exercises 3

Let \( y = f(x) \) be a function defined for \( x \) in the interval \([a, b]\). Fix a positive integer \( n \). For each \( i = 0, 1, 2, \ldots, n \), define

\[
x_i = a + (b - a) \frac{i}{n}
\]

and

\[
\Delta x_i = x_{i+1} - x_i = \left[ a + (b - a) \frac{i+1}{n} \right] - \left[ a + (b - a) \frac{i}{n} \right] = \frac{b - a}{n}.
\]

For each \( i = 1, 2, \ldots, n \) choose a value \( c_i \) in the interval \([x_{i-1}, x_i]\).

The Riemann sum for this data is

\[
\sum_{i=1}^{n} f(c_i) \Delta x_i.
\]

Examples: (1) The left Riemann sum is obtained by choosing \( c_i = x_{i-1} \). It is equal to

\[
\sum_{i=1}^{n} f(x_{i-1}) \Delta x_i.
\]

(2) The right Riemann sum is obtained by choosing \( c_i = x_i \). It is equal to

\[
\sum_{i=1}^{n} f(x_i) \Delta x_i.
\]

(3) The lower Riemann sum is obtained by choosing a value \( c_i \) for which

\[
f(c_i) = \min \{ f(x) : x_{i-1} \leq x \leq x_i \},
\]

i.e., \( f \) takes on its minimum value on the interval \([x_{i-1}, x_i]\) at \( x = c_i \). It is equal to

\[
\sum_{i=1}^{n} \left[ \min_{x_{i-1} \leq x \leq x_i} f(x) \right] \Delta x_i.
\]

(At least one such \( c_i \) exists by the Extreme Value Theorem. It may not be easy to figure out exactly what its value is! There may be more than one point in \([x_{i-1}, x_i]\) where \( f \) takes on its minimum, but the minimum value \( f(c_i) \) is the same for any such point, so the Riemann sum is well-defined.)

(4) The upper Riemann sum is obtained by choosing a value \( c_i \) for which

\[
f(c_i) = \max \{ f(x) : x_{i-1} \leq x \leq x_i \},
\]

i.e., \( f \) takes on its maximum value on the interval \([x_{i-1}, x_i]\) at \( x = c_i \). It is equal to

\[
\sum_{i=1}^{n} \left[ \max_{x_{i-1} \leq x \leq x_i} f(x) \right] \Delta x_i.
\]

Definition 1. The definite integral of \( y = f(x) \) from \( x = a \) to \( x = b \) is the limit as \( n \to \infty \) of any such sequence of Riemann sums, defined with any choice of points \( c_i \). Note: the limit may not exist! If it does, we say that \( f \) is Riemann integrable on this interval. Not all functions are Riemann integrable. We write \( \int_a^b f(x) \, dx \) for the definite integral of \( y = f(x) \) from \( x = a \) to \( x = b \).

If \( f(x) > 0 \) for \( a \leq x \leq b \), \( \int_a^b f(x) \, dx \) computes the area of the region bounded by the graph of \( f \), the \( x \)-axis and the lines \( x = a \) and \( x = b \). For general functions \( f \), \( \int_a^b f(x) \, dx \) computes the signed area between the graph of \( f \) and the \( x \)-axis between \( x = a \) and \( x = b \), as discussed in class.
Exercises
In the first six problems below, let $y = f(x) = 3x + 1$ be defined on the interval $1 \leq x \leq 3$.

1. For a fixed but arbitrary natural number $n$, find formulas for $x_i$ and $\Delta x_i$.

2. Write out the expression for the right Riemann sum $\sum_{i=1}^{n} f(x_i) \Delta x_i$.

3. Evaluate the expression you found in the previous problem. You may need to use the formulas $\sum_{i=1}^{n} 1 = n$ and $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Express your answer in the form $A + \frac{B}{n}$ for some explicit values $A$ and $B$.

4. Show that the answer to the previous problem has a limit as $n \rightarrow \infty$ and compute it.
(5) Sketch the graph of the function $y = f(x) = 3x + 1$ on the interval $1 \leq x \leq 3$ and the region bounded by the graph of $f$, the $x$-axis, and the lines $x = 1$ and $x = 3$. Use elementary plane geometry to find the area of this region. Compare with your answer to the previous problem.

(6) **Uses calculus!** Use the Fundamental Theorem of Calculus to find $\int_{1}^{3} (3x + 1) \, dx$. Compare your answer to the previous two problems.
In these problems, let \( y = f(x) \) be an arbitrary function defined on an arbitrary interval \( a \leq x \leq b \).

(7) **Midpoint Riemann sums.** Let \( c_i \) be the midpoint of \( [x_{i-1}, x_i] \), where \( x_i \) is as on the first page. Find a formula for \( c_i \) and write the corresponding Riemann sum.

(8) **Trapezoid Rule.** Consider the average of the left and right Riemann sums

\[
\frac{1}{2} \left[ \sum_{i=1}^{n} f(x_{i-1}) \Delta x_i + \sum_{i=1}^{n} f(x_i) \Delta x_i \right] = \sum_{i=1}^{n} \left( \frac{f(x_{i-1}) + f(x_i)}{2} \right) \Delta x_i.
\]

Explain why this expression is different from the expression you wrote in the previous problem, for general functions \( f \). For which functions is it the same? Draw a picture to illustrate what this expression is computing.

(Hint: the quantity \( \left( \frac{f(x_{i-1}) + f(x_i)}{2} \right) \Delta x_i \) calculates the area of a trapezoid with two parallel vertical sides of length \( f(x_{i-1}) \) and \( f(x_i) \) at a horizontal distance \( \Delta x_i \) from each other.)