Integration

1. Evaluate using a parametrization of the contour.

\[ I_n = \oint \frac{e^{-2\alpha}}{z^2 - 2z} \, dz = \oint_{|z - 2\alpha| = R} \frac{1}{2\pi i} \, dz, \quad n \neq -1 \]

Solution. \( z(t) = z_0 + R e^{it}, \quad 0 < t < 2\pi, \quad \frac{dz(t)}{dt} = iRe^{it} \, dt \)

\[ I_n = \oint_0^{2\pi} (Re^{it})^n iRe^{it} \, dt = iR^{n+1} \oint_0^{2\pi} e^{i(n+1)t} \, dt \]

\[ I_{-1} = \oint_0^{2\pi} i \, dt = 2\pi i; \quad n \neq -1 \]

\[ I_n = iR^{n+1} \frac{e^{i(n+1)t} \bigg|_0^{2\pi}}{i(n+1)} = 0 \]

2. Evaluate \( I = \oint_0^{\frac{\pi}{2}} \frac{1}{z} \, dz \) along the left semicircle with center at \( 2i \).

Solution. \( z(t) = 2i + 2e^{it}, \quad t \) runs down from \( \frac{3\pi}{2} \) to \( \frac{\pi}{2} \).

\[ z = -2i + 2e^{-it} \quad dz = 2i e^{it} \, dt \]

\[ I = \oint_{\frac{3\pi}{2}}^{\frac{\pi}{2}} (-2i + 2e^{-it}) 2i e^{it} \, dt = \]

\[ = \oint_{\frac{3\pi}{2}}^{\frac{\pi}{2}} (4e^{it} + 4i) \, dt = 4 \frac{e^{it}}{i} \bigg|_{\frac{3\pi}{2}}^{\frac{\pi}{2}} - 4\pi i \]

\[ = \frac{4}{i} (e^{i\pi/2} - e^{-3i\pi/2}) - 4\pi i = \frac{4}{i} (i - (-i)) - 4\pi i \]

\[ = 8 - 4\pi i \]
3. Evaluate \( I = \int_{-8}^{1} z^{1/3} \, dz \) along a semicircle in the upper half-plane, where \( z \sqrt{3} = r \sqrt{3} \, e^{i\theta/3}, \quad z = re^{i\theta}, \quad 0 < \theta < 2\pi \).

**Solution.** Recall if \( a \in \mathbb{R} \), then

\[
(z^a)' = az^{a-1}, \quad \text{where} \quad z = re^{i\theta},
\]

\[
z^a = r^a \, e^{ia\theta}, \quad z^{a-1} = r^{a-1} \, e^{i(a-1)\theta},
\]

and the same branch of \( \theta = \arg z \) is used for both \( z^a \) and \( z^{a-1} \).

\[
I = \int_{-8}^{1} z^{1/3} \, dz = \frac{3}{4} \left[ \left. \frac{z^{4/3}}{4/3} \right|_8^1 \right] = \frac{3}{4} \left( 1^{4/3} - (-8)^{4/3} \right)
\]

\[
1^{4/3} = 1 - \text{limiting value (} \theta > 0 \))
\]

\[
(-8)^{4/3} = 8^{4/3} \, e^{i\pi \cdot 4/3} = 16 \, \frac{-1-i\sqrt{3}}{2}
\]

\[
I = \frac{3}{4} \left( 1 - 16 \, \frac{-1-i\sqrt{3}}{2} \right) = \frac{3}{4} \left( 9 + 8\sqrt{3} \, i \right)
\]

4. Estimate \( I = \int_{i}^{1} \frac{dz}{z^4} \) along the straight line.

**Solution** \( |I| \leq \max \left| \frac{1}{z^4} \right| \), length =

\[
= \frac{1}{(\min |z|)^4} \cdot 2^2 = \frac{1}{(\frac{1}{2})^4} \cdot 2 = 4\sqrt{2}
\]
Evaluate \( I = \int_0^{\infty} e^{-x^2} \cos(2b x) \, dx \), \( b > 0 \), assuming \( \int_0^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \).

**Solution.** Note \( I = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} \cos(2b x) \, dx \) (\( \cos \) is even!)

Then \( I = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} (\cos(2b x) - i \sin(2b x)) \, dx \) (\( \sin \) is odd!)

Then \( I = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2 - 2ibx} \, dx \)

Completing the square:

\[
-x^2 - 2ibx = -(x^2 + 2ibx - b^2 + b^2) = -(x + ib)^2 - b^2
\]

Then \( I = \frac{1}{2} \int_{-\infty}^{\infty} e^{-(x + ib)^2} e^{-b^2} \, dx = \frac{1}{2} e^{-b^2} \lim_{a \to \infty} \int_{-a+ib}^{a+ib} e^{-z^2} \, dz \)

By the Cauchy thm \( \int_{-a+ib}^{a+ib} e^{-z^2} \, dz = \int_{a}^{a} + \int_{-a}^{-a} - \int_{-a+ib}^{-a+ib} \)

On the line \([a, a+ib]\\):

\[
|e^{-z^2}| = e^{-Re(z^2)} = e^{y^2 - x^2} \leq e^{b^2 - a^2}
\]

\[
\left| \int_{a}^{a+ib} e^{-z^2} \, dz \right| \leq e^{b^2 - a^2} \cdot b \to 0 \text{ as } a \to \infty
\]

Similarly \( \left| \int_{-a}^{-a+ib} e^{-z^2} \, dz \right| \to 0 \text{ as } a \to \infty \)

Hence \( I = \frac{1}{2} e^{-b^2} \lim_{a \to +\infty} \int_{-a}^{a} e^{-z^2} \, dz = \frac{1}{2} e^{-b^2} \int_{-\infty}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \)
6. Evaluate \( I = \int_{|z|=5} \frac{z \, dz}{(z-1)(z+2)} \)

**Solution:** \( I = \int_{|z|=5} \frac{(z+1) - 1}{(z-1)(z+2)} \, dz \)

\[
I = \int_{|z|=5} \frac{dz}{z+2} - \int_{|z|=5} \frac{dz}{(z-1)(z+2)}
\]

Consider \( F(R) = \int_{|z|=R} \frac{dz}{(z-1)(z+2)} \), \( R > 2 \).

Then \( F(R) \) is independent of \( R \) by the Cauchy thm.

\[
|F(R)| \leq \frac{2\pi R}{(R-1)(R-2)} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty.
\]

Hence \( F(R) = 0 \), \( R > 2 \).

Hence \( I = \int_{|z|=5} \frac{dz}{z+2} = \int_{|z+2|=1} \frac{dz}{z+2} = 2\pi i \)


Partial fractions: \( \frac{z}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2} \)

\( z = A(z+2) + B(z-1) \)

Plug in \( z = -2 \) \( \Rightarrow B = \frac{2}{3} \)

\( z = 1 \) \( \Rightarrow A = \frac{1}{3} \)

\[
I = \frac{1}{3} \int_{|z|=5} \frac{dz}{z-1} + \frac{2}{3} \int_{|z|=5} \frac{dz}{z+2} =
\]

\[
= \frac{1}{3} \int_{|z-1|=1} \frac{dz}{z-1} + \frac{2}{3} \int_{|z+2|=1} \frac{dz}{z+2} =
\]

\[
= \frac{1}{3} (2\pi i) + \frac{2}{3} (2\pi i) = 2\pi i
\]
8. Yet another solution of #6!

By the Cauchy thm:

\[ I = \int_{|z-1|=1} \frac{z}{(z-1)(z+2)} \, dz + \int_{|z+2|=1} \frac{z}{(z-1)(z+2)} \, dz = 2\pi i \left[ \frac{z}{z+2} \right]_{z=1}^{z=-2} = \frac{2\pi i}{3} + \frac{4\pi i}{3} = \frac{2\pi i}{3} \]

9. Evaluate \( I = \int_{|z|=2} \frac{e^{\pi z/2}}{(z+i)^3} \, dz \)

Solution \( f(z) = e^{\pi z/2} \) is analytic in \( |z| \leq 2 \).

\[ I = \int_{|z|=2} \frac{f(z)}{(z+i)^3} \, dz \]

By the Cauchy formula for derivatives:

\[ I = \frac{2\pi i}{2!} f''(-i) \]

\[ f''(z) = \frac{\pi^2}{4} e^{\pi z/2} \]

\[ I = \frac{\pi^2}{4} e^{-\pi i/2} = i \frac{\pi^3}{4} (-i) = \frac{\pi^3}{4} \]