\[ I = \oint_{|z+1|=2} \frac{z^2}{y-z^2} \, dz \]

\[ y-z^2 = (2-z)(2+z) \]

vanishes at \( z = \pm 2 \).

-2 is inside the contour,

2 is outside.

\[ f(z) = \frac{z^2}{2-z} \] is analytic inside and on the contour.

By the Cauchy formula

\[ I = \oint_{|z+1|=2} \frac{f(z)}{z+2} \, dz = \frac{2 \pi i}{z+2} f(-2) \]

\[ = 2 \pi i \frac{(-2)^2}{2-(-2)} = \boxed{2 \pi i} \]
\[ \Gamma = \int_{|z+i|=2} \frac{dz}{(z^2 + 4)^3} \]

\[ z^2 + 4 = (z - 2i)(z + 2i) \]

\[ f(z) = \frac{1}{(z - 2i)^3} \quad \text{analytic inside/on } \gamma. \]

\[ \Gamma = \int_{\gamma} \frac{f(z)}{(z + 2i)^3} dz = \frac{2\pi i}{2!} f''(-2i) \]

\[ f''(z) = \frac{12}{(z - 2i)^5} \]

\[ \Gamma = \frac{\pi i \cdot 12}{(-2i - 2i)^5} = -\frac{12\pi}{4^5} \]

\[ \Gamma = -\frac{3\pi}{2^5 6} \]
\[ I = \int \frac{\log z \, dz}{(z^2 + 4)^2} \]

Branch cut of the Log
\[ y = \{ \text{1, 3, 1 = 2} \}, \text{ counter clockwise} \]

Note \((z^2 + 4)^2 = (z - 2i)^2(z + 2i)^2\)

2i is outside \(Y\)

-2i is inside \(Y\)

\[ f(z) = \frac{\log z}{(z - 2i)^2} \]

is analytic inside and on \(Y\).

By the Cauchy formula (for the derivative)

\[ I = \int \frac{f(z) \, dz}{(z + 2i)^2} = \frac{2\pi i}{1!} f'(-2i) \]

\[ f'(z) = \frac{1}{z(2z - 2i)^2} - \frac{2\log z}{(2z - 2i)^3} \]

\[ f'(-2i) = \frac{1}{-2i(-4i)^2} - \frac{2\log(-2i)}{(-4i)^3} \]

\[ = \frac{1}{32i} - \frac{\ln 2 - \frac{\pi i}{2}}{32i} = \frac{1}{32i} \left( 1 - \ln 2 + \frac{\pi i}{2} \right) \]

\[ I = \frac{\pi}{16} \left( 1 - \ln 2 + \frac{\pi i}{2} \right) \]
4. (Problem 2.3 # 13) Integrate $e^{iz^2}$ around the contour $\gamma$ shown in Figure 2.8 (a $\frac{1}{8}$th of a pie) to obtain the Fresnel integrals

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

By Cauchy’s Theorem since $e^{iz^2}$ is entire, we have that $\int_\gamma e^{iz^2}dz = 0$

Also, we have that

$$\int_{\gamma_1} e^{iz^2}dz + \int_{\gamma_2} e^{iz^2}dz + \int_{\gamma_3} e^{iz^2}dz = \int_\gamma e^{iz^2}dz = 0,$$

where $\gamma_1(t) = t$ & $t$ ranges from 0 to $R$, $\gamma_2(t) = Re^{it}$ & $t$ ranges from 0 to $\frac{\pi i}{4}$, and $\gamma_3(t) = te^{\pi i}t$ & $t$ ranges from $R$ to 0

Note that

$$|\int_{\gamma_2} e^{iz^2}dz| \leq \int_{\gamma_2} |e^{iz^2}dz| \leq \int_0^{\frac{\pi}{4}} Re^{-R^2\sin(2t)}dt \leq \int_0^{\frac{\pi}{4}} Re^{-R^2\frac{4t}{\pi}}dt$$

(since $\sin(2t) \geq \frac{4t}{\pi}$ for $t \in [0, \frac{\pi}{4}]$)

$$= \frac{\pi}{4R}e^{-R^2\frac{4t}{\pi}}\bigg|_0^{\frac{\pi}{4}} = \frac{\pi}{4R}(e^{-R^2} - 1) \to 0 \text{ as } R \to \infty$$

Also,
\[
\int_{\gamma_3} e^{iz^2} \, dz = \int_0^R e^{it^2} e^{i\pi\over 4} e^{i\pi t} \, dt
\]
\[
= \frac{1 + i}{\sqrt{2}} \int_0^R e^{-t^2} \, dt
\]
\[
\to \quad -\frac{1 + i}{\sqrt{2}} \int_0^\infty e^{-t^2} \, dt
\]
\[
= \frac{1 + i}{\sqrt{2}} \left( \frac{\sqrt{\pi}}{2} \right)
\]
\[
= \frac{\sqrt{2\pi}}{4} (1 + i),
\]
as \(R \to \infty\), and making use of \(\int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2}\).

This implies that
\[
\int_0^\infty e^{it^2} \, dt = \frac{\sqrt{2\pi}}{4} (1 + i)
\]
\[
\int_0^\infty \cos(t^2) \, dt + i \int_0^\infty \sin(t^2) \, dt = \frac{\sqrt{2\pi}}{4} (1 + i)
\]
\[
\Rightarrow \int_0^\infty \cos(t^2) \, dt = \int_0^\infty \sin(t^2) \, dt = \frac{\sqrt{2\pi}}{4}
\]