F(z) is analytic on \(|z - z_0| < R\),
\(\text{Re} \ F(z_0) = 0\), \(0 < r < R\); \(F \neq \text{const}\).
Prove \(\text{Re} \ F\) takes both positive & negative values on \(|z - z_0| = r\).

Proof. Put \(u = \text{Re} \ F\). By the mean-value theorem,
\[
0 = u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it}) \, dt
\]
If, say, \(u(z_0 + re^{it}) > 0\) for all \(t\), then
since \(u\) is continuous, then \(u(z_0 + re^{it}) = 0\)
for all \(t\).
So either \(u\) takes both \(\pm\) values on
\(|z - z_0| = r\) or \(u \equiv 0\) on \(|z - z_0| = 0\).
The latter is not the case b.c.

by the Maximum Principle, \(u(z)\) for \(|z - z_0| \leq r\)
takes both maximum and minimum values
on the circle \(|z - z_0| = r\), so \(u \equiv 0\) in \(|z - z_0| \leq r\).
But then \(F \equiv \text{const} \implies c = 0\).
3.2 # 9

\( f \in \mathbb{A}(D), \quad D \text{ - domain} \)

\( \gamma \) simple closed curve \( \gamma \subset D \)

let \( |f| = \text{const} \) on \( \gamma \).

Then either

1) \( f = \text{const} \) on \( D \)

or 2) \( f \) has a zero inside \( \gamma \).

**Solution**

Suppose \( f \) has no zeros inside \( \gamma \).

Then \( f \) satisfies the minimum principle inside \( \gamma \), i.e., \( \frac{1}{f} \) satisfies the maximum principle inside \( \gamma \).

Since \( |f| \) attains both minimum and maximum on \( \gamma \), then \( |f| = \text{const} \) inside \( \gamma \). Then \( f = \text{const} \) inside \( \gamma \), whence in \( D \).

So if 2) does not hold,

then 1) holds.

**DONE**
3.2 #12

\[ f \in A(D), \quad D = \{ z \in \mathbb{C} : |z| < 1 \} \]

\[ |f(z)| \leq M \quad \text{for} \quad |z| < 1. \]

Let \( f(z) = 0 \) for some \( z \in D. \)

Show \( |f(z)| \leq M \left| \frac{z - \alpha}{1 - \bar{z} z} \right|, \quad |z| < 1. \)

**Solution**

Observe that for \( |z| = 1, \)

\[ \left| \frac{z - \alpha}{1 - \bar{z} z} \right| = \frac{|z - \alpha|}{|z| \left| \frac{1}{z} - \bar{z} \right|} = \frac{|z - \alpha|}{|z - \bar{z}|} = 1 \]

Since \( \frac{z - \alpha}{1 - \bar{z} z} \) is continuous (\( z \neq \frac{1}{z} \))

Then \( \min_{|z| = r} \left| \frac{z - \alpha}{1 - \bar{z} z} \right| \rightarrow 1 \) as \( r \rightarrow 1 \)

Let \( g(z) = \frac{f(z)}{(1 - \bar{z} z)} \)

Since \( f(z) = 0, \) then \( g \) has a removable singularity at \( \alpha. \)

Let \( |z_0| < r < 1. \) By Max. Principle

\[ |g(z_0)| \leq \max_{|z| = r} |g(z)| \leq \frac{M}{\min_{|z| = r} \left| \frac{z - \alpha}{1 - \bar{z} z} \right|} \rightarrow M \quad \text{as} \quad r \rightarrow 1. \]

Hence \( |g(z_0)| \leq M \) for all \( z_0 \in D. \)

Hence \( |f(z)| \leq M \left| \frac{z - \alpha}{1 - \bar{z} z} \right| \)

DONE!
Let \( f \) be analytic in \(|z| < 1\), and let \(|f(z)| \leq 1\) for all \(|z| < 1\). Let \( f(0) = 0\).

Prove:

a) \(|f'(0)| \leq 1\)

b) If \(|f'(0)| = 1\), then \( f(z) = cz\) for some \( c \in \mathbb{C}, |c| = 1\).

Proof:

Since \( 0 \) is a zero of \( f \), then \( f(z) = z \cdot g(z) \), where \( g \) is analytic in \(|z| < 1\).

Apply the Max. Principle to \( g \) in the disk \(|z| \leq r, 0 < r < 1\).

For \(|z| = r\), \( |g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r} \).

Hence, for every \(|z| < 1\) and every \( r > |z|\), \(|g(z)| \leq \frac{1}{r} \to 1\) as \( r \to 1\).

So \(|g(z)| \leq 1\) for all \(|z| < 1\), in particular \(|g(0)| \leq 1\).

\[ f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} g(z) = g(0) \]

Hence \(|f'(0)| = |g(0)| \leq 1\). \( \Rightarrow a)\).

Let \(|f'(0)| = 1\). Then \(|g|\) has a maximum at \( 0 \). Hence \( g = \text{const} = c\), \(|c| = 1\).

Then \( f(z) = z \cdot g(z) = cz \Rightarrow b) \)

DONE!