\[ f(z) = z^4 - 3z^2 + 3 \]

\# zeros in the 1st quadrant = ?

\[ C = [0, R] + [iR, 0] + C_R \]

We’ll see \( f \) has no zeros on \( C \) for \( R \gg 1 \).

Then \( Z_f = \frac{1}{2\pi} \Delta_C \arg f(z) \)

\[ a) [0, R] \quad z = x > 0 \]
\[ f(z) = x^4 - 3x^2 + 3 = t^2 - 3t + 3, \quad t = x^2 \]
\[ t^2 - 3t + 3 = 0 \text{ has no real roots} \]

Hence \( f(z) > 0 \).

\[ \Delta_{[0,R]} \arg f(z) = 0 \]

\[ b) [iR, 0] \quad z = iy \quad y > 0 \]
\[ f(z) = y^4 + 3y^2 + 3 > 0 \text{ again!} \]
\[ \Delta_{[iR,0]} \arg f(z) = 0 \]

\[ c) C_R \quad f(z) = z^4 g(z) \quad g(z) = 1 - \frac{3}{z^2} + \frac{3}{z} \]

\[ \Delta_{C_R} \arg f(z) = 4 \Delta_{C_R} \arg z + \Delta_{C_R} \arg g(z) \]

Since \( g(z) \approx 1 \) for \( R \gg 1 \), then

\[ \Delta_{C_R} \arg f(z) = 4 \cdot \frac{\pi}{2} + 3 \quad (|\varepsilon| << 1) \]

\[ \Delta_{C} \arg f(z) = 2\pi + \varepsilon \]

Since \( Z_f = \frac{\Delta_C \arg f(z)}{2\pi} \text{ is integer,} \]

\[ Z_f = 1 \]
\[ f(z) = 2z^4 - 2iz^3 + z^2 + 2iz - 1 \]

# zeros in \( \text{Im} z > 0 \)?

\[ Z_f = \frac{1}{2\pi} \Delta \arg \frac{1}{z} \]

\[ C = [-R, R] + C_R, \quad R \gg 1 \]

\[ C_R : \quad \Delta_{C_R} \arg f \approx 4 \Delta_{C} \arg z = 4\pi \]

\([-R, R] : \quad z = x \in \mathbb{R} \]

\[ \text{Re} f(x) = 2x^4 + x^2 - 1 = 0 \quad x^2 = -1 \pm \sqrt{1 + 8} = -\frac{1}{2}, \begin{cases} \frac{3}{2} \end{cases} \]

\[ \text{two roots} \quad x = \pm \frac{\sqrt{3}}{2} \]

\[ \text{Im} f(x) = 2x(1 - x^2) = 0 \quad \text{has 3 roots} \quad 0, \pm 1 \]

Let's use \( \text{Re} f(x) \) to define branches of \( \arg f(x) \).

\[ \text{Sign of } \text{Re} f(x) : -R \quad L_1 \quad -\frac{\sqrt{3}}{2} \quad L_2 \quad \frac{\sqrt{3}}{2} \quad L_3 \quad R \]

\[ f(\pm \frac{\sqrt{3}}{2}) = \pm \frac{i\sqrt{3}}{2} \]

\[ f(\pm R) = 2R^2 > 0 \]

\[ L_1 : \quad \text{Re} f(x) > 0 \quad -\frac{\pi}{2} \leq \arg f(x) \leq \frac{\pi}{2} \]

\[ \Delta L_1 \arg f(x) = \arg f\left(\frac{\sqrt{3}}{2}\right) - \arg f\left(-\frac{\sqrt{3}}{2}\right) \approx -\frac{\pi}{2} \quad -0 = -\frac{\pi}{2} \]

\[ L_3 : \quad \text{Similar} \]

\[ \Delta L_3 \arg f(x) = \arg f\left(\frac{\sqrt{3}}{2}\right) - \arg f\left(-\frac{\sqrt{3}}{2}\right) \approx \frac{\pi}{2} - \frac{3\pi}{2} = -\pi \]

\[ L_2 : \quad \text{Re} f(x) \leq 0 \quad \frac{\pi}{2} \leq \arg f(x) \leq \frac{3\pi}{2} \]

\[ \Delta L_2 \arg f(x) = \arg f\left(\frac{\sqrt{3}}{2}\right) - \arg f\left(-\frac{\sqrt{3}}{2}\right) = \frac{\pi}{2} - \frac{3\pi}{2} = -\pi \]

\[ \Xi_f = \frac{1}{2\pi} \left(4\pi - \frac{\pi}{2} - \frac{3\pi}{2} - \pi\right) = 1 \]

\[ \Xi_f = 1 \]
3.1 #10
Show there is no entire function $F$ with $F(x) = 1 - e^{2\pi i / x}$ for $1 \leq x \leq 2$.

Assume such function $F(z)$ exists. Let $G(z) = 1 - e^{2\pi i / z}$.

The functions $F$ and $G$ are analytic in the domain $\mathbb{C} \setminus \{0\}$.

They are equal on the set $[1, 2] \cap \mathbb{R} \cap \mathbb{C}$ whose points are not isolated.

By the uniqueness (identity) theorem $F = G$ in $\mathbb{C} \setminus \{0\}$.

Since $F$ is entire, then

$$\lim_{z \to 0} F(z) = \lim_{z \to 0} G(z) \text{ exist.}$$

But $G$ has an essential singularity at $0$ so $\lim_{z \to 0} G(z)$ does not exist.

$\rightarrow \leftarrow$. 
3. (Problem 3.1 #12) Find the number of zeroes of \( z^3 - 3z + 1 \) in the annulus \( 1 < |Z| < 2 \)

When \( |z| = 1 \), we have that

\[
| (z^3 - 3z + 1) + 3z | = |z^3 + 1| \\
\leq |z|^3 + 1 \\
= 2 \\
\leq 3 = |3z|
\]

so by Rouche’s Theorem, \( z^3 - 3z + 1 \) has the same number of zeroes in the disk bounded by \( |z| = 1 \) as \( 3z \). Since \( 3z \) has 1 zero in this region, \( z^3 - 3z + 1 \) has 1 zero in this disk.

When \( |z| = 2 \), we have that

\[
| (z^3 - 3z + 1) - z^3 | = |3z - 1| \\
\leq 3|z| + 1 \\
= 6 + 1 \\
\leq 8 = |z^3|
\]

so by Rouche’s Theorem, \( z^3 - 3z + 1 \) has the same number of zeroes in the disk bounded by \( |z| = 2 \) as \( z^3 \), which has 3 zeroes in this region, counting multiplicity. This gives us that \( z^3 - 3z + 1 \) has 3 zeroes in this region.
By subtracting, we get that $z^3 - 3z + 1$ has $3 - 1 = 2$ zeroes in the annulus $1 < |z| < 2$. 