1. (5 points) Evaluate

(a) (2 pts) \( \log(-\sqrt{3} + i) \) (Write all values in the form \( a + ib \) with real \( a \) and \( b \). Trigonometric or inverse trigonometric functions are not allowed.)

\[
z = -\sqrt{3} + i \quad |z| = 2
\]

\[
z = 2 e^{\frac{5\pi i}{6}}
\]

\[
\log z = \ln |z| + i \arg z
\]

\[
= \ln 2 + \frac{5\pi i}{6} + 2\pi i n
\]

\( n \in \mathbb{Z} \)

(b) (3 pts) \( \sqrt[3]{8i} \) (Write all values in the form \( a + ib \) with real \( a \) and \( b \). Trigonometric or inverse trigonometric functions are not allowed.)

\[
z = 8i = 8 e^{\frac{i\pi}{2}}
\]

\[
3\sqrt[3]{z} = 2 e^{\frac{i\pi}{6} + \frac{2\pi i n}{3}}, \quad n = 0, 1, 2.
\]

\( n=0 \) \( z_0 = 2 e^{\frac{i\pi}{6}} = \sqrt{3} + i \)

\( n=1 \) \( z_1 = 2 e^{\frac{5i\pi}{6}} = -\sqrt{3} + i \)

\( n=2 \) \( z_2 = 2 e^{\frac{3i\pi}{2}} = -2i \)
2. (5 points) Describe and sketch the image of the line \( \{ z \in \mathbb{C} : \text{Im} z = 1 \} \) under the map \( f: \mathbb{C} \setminus \{0\} \to \mathbb{C} \) given by \( f(z) = 1/z \).

Let \( z \in \mathbb{L} \), \( \frac{z - \overline{z}}{2i} = 1 \), \( z - \overline{z} = 2i \).

Let \( w = f(z) = \frac{1}{z} \), \( z = \frac{1}{w} \).

Then \( \frac{1}{w} - \frac{1}{\overline{w}} = 2i \), \( \overline{w} - w = 2i \).

\( w\overline{w} + \frac{i}{2} \overline{w} - \frac{i}{2} w = 0 \)

Note \( |w - a|^2 = (w - a)(\overline{w} - \overline{a}) = w\overline{w} - a\overline{w} - \overline{a}w + aa \)

Put \( a = -\frac{i}{2} \), \( a\overline{a} = \frac{1}{4} \)

\( w\overline{w} + \frac{i}{2} \overline{w} - \frac{i}{2} w + \frac{1}{4} = \frac{1}{4} \)

\( |w + \frac{i}{2}|^2 = \frac{1}{4} \)

Hence \( \mathbb{w} \in \mathbb{K} = \{ w \in \mathbb{C} : |w + \frac{i}{2}| = \frac{1}{2} \} \)

\( \circ \) circle of radius \( \frac{1}{2} \) with center at \( -\frac{i}{2} \).

Conversely, if \( w \neq 0 \) \( w \in \mathbb{K} \), then \( z = \frac{1}{w} \).

By going back, we see \( z \in \mathbb{L} \).

Hence \( f(\mathbb{L}) = \mathbb{K} \setminus \{0\} \).
3. (5 points) Derive an expression of \( \text{arccosh} \, z \) in terms of \( \log \) and \( \sqrt{\_} \), where \( \text{arccosh} \, z \) is the inverse function to \( \cosh \, z = (e^z + e^{-z})/2 \).

\[
\begin{align*}
  w &= \text{arccosh} \, z \\
  z &= \cosh \, w = \frac{e^w + e^{-w}}{2} \\
  e^w + e^{-w} - 2z &= 0 \\
  e^{2w} - 2ze^w + 1 &= 0 \\
  e^w &= z \pm \sqrt{z^2 - 1} \\
  w &= \log \left( z \pm \sqrt{z^2 - 1} \right)
\end{align*}
\]
4. (5 points) Show that if \( f \) is an entire (analytic in the whole plane) function, and \( \text{Im} f(z) = (\text{Re} f(z))^2 \) for all \( z \in \mathbb{C} \), then \( f \) is constant.

\[
\text{Let } f = u + iv, \quad u, v \text{ - real}
\]

Then \( v = u^2 \)

Take \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) of both sides.

1. \( V_x = 2uu_x \)
2. \( V_y = 2uu_y \)

Using Cauchy–Riemann equations

3. \( u_x = v_y \)
4. \( u_y = -v_x \)

Express \( V_x \) and \( V_y \) in terms of \( u_x \) and \( u_y \)

Then (1) and (2) will turn into

5. \( -u_y = 2uu_x \)
6. \( u_x = 2uu_y \)

Plug \( u_y \) from (5) into (6)

\[
\begin{align*}
\text{plugging } u_y \text{ from (5)} & \quad \text{into (6)} \\
\Rightarrow u_x & = 2u(-2uu_x) \\
(1 + 4u^2)u_x & = 0 \\
1 + 4u^2 > 0 \quad \Rightarrow \quad u_x = 0
\end{align*}
\]

Now by (5) \( u_y = 0 \)

and by (3 - 4) \( V_x = V_y = 0 \)

Hence \( u = \text{const} \quad v = \text{const} \)

\[
\hat{f} = \text{const}
\]
5. (5 points) Find power series expansions $\sum_{n=0}^{\infty} a_n(z+2)^n$ of the following functions. In both cases express the coefficients $a_n$ of the series in terms of $n$.

(a) (3.5 pts) $(z+7)^{-1}$

\[
\frac{1}{z+7} = \frac{1}{(z+2) + 5} = \frac{1}{5\left[1 - \left(-\frac{z+2}{5}\right)\right]} \\
= \frac{1}{5} \sum_{n=0}^{\infty} \left(-\frac{z+2}{5}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(z+2)^n}{5^{n+1}}
\]

\[a_n = \frac{(-1)^n}{5^{n+1}}, \quad n = 0, 1, 2, \ldots\]

(b) (1.5 pts) $(z+7)^{-3}$

\[
\left(\frac{1}{z+7}\right)' = -\frac{1}{(z+7)^2} \quad \left(\frac{1}{z+7}\right)'' = \frac{2}{(z+7)^3}
\]

\[
\frac{1}{(z+7)^3} = \frac{1}{2} \left(\frac{1}{z+1}\right)''' = \text{differentiate (a)}
\]

\[
= \sum_{n=2}^{\infty} \frac{(-1)^n \cdot n(n-1)(z+7)^{n-2}}{2 \cdot 5^{n+1}} = \text{shift index}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)(z+7)^n}{2 \cdot 5^{n+3}}
\]

\[a_n = \frac{(-1)^n(n+1)(n+2)}{2 \cdot 5^{n+3}}, \quad n = 0, 1, 2, \ldots\]
6. (5 points) Let \( f = u + iv \) be a function defined in a neighborhood of \( z_0 \in \mathbb{C} \). Suppose the derivative \( f'(z_0) \) exists. Prove that the partial derivatives \( u_x, u_y, v_x, v_y \) exist and satisfy the Cauchy-Riemann equations at \( z_0 \).

Evaluate the limit \( f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \) in two different ways:

Along the line \( y = y_0 \):
(here \( z = x + iy, \ z_0 = x_0 + iy_0, \ f(z) = f(x, y) \))
\[
f'(z_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} =: f_x(z_0)
\]

Along the line \( x = x_0 \):
\[
f'(z_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} =: -i f_y(z_0)
\]

Hence, the partials \( f_x \) and \( f_y \) exist at \( z_0 \) and satisfy \( f_x = -i f_y \).

The partials \( u_x, u_y, v_x, v_y \) exist at \( z_0 \) as components of the vectors \( f_x, f_y \). We have
\[
u_x + iv_x = -i(u_y + iv_y)
\]
By separating the real and imaginary parts,
\[
\begin{align*}
u_x &= v_y, & v_x &= -u_y
\end{align*}
\]
which are the Cauchy-Riemann equations.