Theorem Let \( f \) be a continuous function in a nbd. of \( z_0 \in \mathbb{C} \). Let \( f'(z_0) \) exists and \( f'(z_0) \neq 0 \). Let \( g \) be a continuous branch of \( f^{-1} \) in a nbd of \( w_0 = f(z_0) \) such that \( g(w_0) = z_0 \), that is, \( f(g(w)) = w \) for all \( w \) in a nbd of \( w_0 \).

Then \( g'(w_0) \) exists and \( g'(w_0) = \frac{1}{f'(z_0)} \)

Proof. Define \( F(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) , & z \neq z_0 \\ 0 , & z = z_0 \end{cases} \)

Then \( F \) is continuous at \( z_0 \), and

\[
 f(z) - f(z_0) = (f'(z_0) + F(z))(z - z_0)
\]

Put \( z = g(w) \) for \( w \) in a nbd of \( w_0 \):

\[
 f(g(w)) - w_0 = (f'(z_0) + F(g(w)))(g(w) - g(w_0))
\]

\[
 \frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{f'(z_0) + F(g(w))}
\]

Passing to the limit as \( w \to w_0 \):

\[
 g'(w_0) = \lim_{w \to w_0} \frac{1}{f'(z_0) + F(g(w))} = \frac{1}{f'(z_0)}
\]

DONE!
Remarks

1. The above proof is a word by word repetition of the one for real functions of one real variable, but the meaning of the theorem is different b.c. it involves operations with complex numbers.

2. The above theorem is somewhat incomplete. We will see later that if $f'(z_0) \neq 0$, then the inverse function $g$ exists automatically. Conversely, if $f'(z_0) = 0$, then there is no continuous $g$ with stated properties. The converse is essentially complex variable statement, because it does not hold in real analysis.

3. Nevertheless, this theorem suffices to conclude that continuous branches of the inverse functions of analytic functions are analytic. Hence, continuous branches of log $z$ and inverse trig. functions are analytic.
\( \text{R-differentiable functions of 2 variables} \)

Recall the following result from Calculus 3 was now with proof!

**Thm.** Let \( U(x,y) \) be a real function with partial derivatives \( U_x(x,y) \) and \( U_y(x,y) \) in a nbd. of \((x_0,y_0) \in \mathbb{R}^2\). Suppose \( U_x \) and \( U_y \) are continuous at \((x_0,y_0)\). Then \( U \) is differentiable at \((x_0,y_0)\) that is \( U \) has a linear approximation:

\[
U(x,y) = U(x_0,y_0) + U_x(x_0,y_0) \Delta x + U_y(x_0,y_0) \Delta y + \\
+ \varepsilon_1(x,y) \Delta x + \varepsilon_2(x,y) \Delta y
\]

so that \( \varepsilon_1, \varepsilon_2 \to 0 \) as \((x,y) \to (x_0,y_0)\)

Here \( \Delta x := x-x_0 \), \( \Delta y := y-y_0 \).

**Proof.** To compare \( U(x,y) \) with \( U(x_0,y_0) \), consider an auxiliary point \((x,y_0)\):

\[
U(x,y) - U(x_0,y_0) = [U(x,y_0)-U(x_0,y_0)] + [U(x,y)-U(x,y_0)]
\]
By the Intermediate Value Theorem

\[ u(x, y) - u(x_0, y_0) = u_x(\xi, y_0)(x - x_0) \]
\[ u(x, y) - u(x, y_0) = u_y(x, \eta)(y - y_0) \]

Here \( \xi \) is between \( x \) and \( x_0 \) and \( \eta \) is between \( y \) and \( y_0 \).

In fact \( \xi = \xi(x) \) depends on \( x \),
and \( \eta = \eta(x, y) \) depends on \( x \) and \( y \).

Put \( \varepsilon_1(x, y) = u_x(\xi, y_0) - u_x(x_0, y_0) \)
\( \varepsilon_2(x, y) = u_y(x, \eta) - u_y(x_0, y_0) \)

Then we have

\[ u(x, y) = u(x_0, y_0) + u_x(x_0, y_0) \Delta x + u_y(x_0, y_0) \Delta y \]
\[ + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad \text{(Check!)} \]

Since \( \xi \to x_0 \) and \( \eta \to y_0 \) as \( (x, y) \to (x_0, y_0) \)
and since \( u_x \) and \( u_y \) are continuous
at \( (x_0, y_0) \), then

\( \varepsilon_1 \) and \( \varepsilon_2 \to 0 \) as \( (x, y) \to (x_0, y_0) \)

as desired. \( \text{DONE!} \)

Remark. Without the hypothesis that
\( u_x \) and \( u_y \) be continuous at \( (x_0, y_0) \)
the conclusion fails in general.
Cauchy–Riemann (C–R) equations imply C–differentiability.

Thm. Let $f$ have partial derivatives $f_x$ and $f_y$ in a nbhd. of $z_0 \in \mathbb{C}$.
Suppose $f_x$ and $f_y$ are continuous at $z_0$ and satisfy the C–R equations
$\bar{f}_x + if_y = 0$ \hspace{1em} ($u_x = v_y$, $u_y = -v_x$) at $z_0$.
Then $\exists f'(z_0) = f_x(z_0) = -i f_y(z_0)$.

Proof. Since $f_x$ and $f_y$ are continuous at $z_0$, $f$ is R–differentiable at $z_0$.
Then
$\Delta f = f(z_0 + \Delta z) - f(z_0)
= f_x(z_0) \Delta x + f_y(z_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$,
here $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta z \to 0$.

Plug in $\Delta x = \frac{\Delta z + \bar{\Delta z}}{2}, \Delta y = \frac{\Delta z - \bar{\Delta z}}{2i}$
Then
$\Delta f = f_x(z_0) \frac{\Delta z + \bar{\Delta z}}{2} + f_y(z_0) \frac{\Delta z - \bar{\Delta z}}{2i} + \\
\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$. 
\[
\Delta f = \frac{1}{2} \left( f_x(z_0) - i f_y(z_0) \right) \Delta z + \frac{1}{2} \left( f_x(z_0) + i f_y(z_0) \right) \overline{\Delta z} = 0 \quad \text{by C-R,}
\]
\[+ \quad \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.\]

\[
\frac{\Delta f}{\Delta z} = \frac{1}{2} \left( f_x(z_0) - i f_y(z_0) \right) + \varepsilon_1 \frac{\Delta x}{\Delta z} + \varepsilon_2 \frac{\Delta y}{\Delta z}
\]

Hence \( \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \frac{1}{2} \left( f_x(z_0) - i f_y(z_0) \right) \) exists.

because \( \varepsilon_1 \to 0, \varepsilon_2 \to 0, \quad \left| \frac{\Delta x}{\Delta z} \right| \leq 1, \quad \left| \frac{\Delta y}{\Delta z} \right| \leq 1.\)

By C-R
\[f'(z_0) = f_x(z_0) = -i f_y(z_0).\]

Example \( f(z) = e^z = e^x (\cos y + i \sin y) \)
\[u = e^x \cos y \quad \text{and} \quad v = e^x \sin y \]
have continuous derivatives
\[u_x = e^x \cos y, \quad u_y = -e^x \sin y, \]
\[v_x = e^x \sin y, \quad v_y = e^x \cos y, \]
that satisfy \( u_x = v_y, u_y = -v_x.\)
Hence \( f'(z) = f_x(z) = e^z \) exists;
\( f \in A(\mathbb{C}) \) is an entire function.
Complex form of the differential

For a function \( f(z) = f(x,y) \),
\[
df := f_x \, dx + f_y \, dy.
\]

Using \( dz = dx + i \, dy \), we have
\[
df = f_x \, \frac{dz + d\overline{z}}{2} + f_y \, \frac{dz - d\overline{z}}{2i}
\]
\[
= \frac{i}{2} (f_x - if_y) \, dz + \frac{1}{2} (f_x + if_y) \, d\overline{z}.
\]

Define
\[
\frac{df}{dz} = \frac{\partial f}{\partial z} := \frac{i}{2} (f_x - if_y)
\]
\[
\frac{df}{d\overline{z}} = \frac{\partial f}{\partial \overline{z}} := \frac{i}{2} (f_x + if_y)
\]

Then
\[
df = \frac{df}{dz} \, dz + \frac{df}{d\overline{z}} \, d\overline{z},
\]
which is the complex form of the differential.

The Cauchy–Riemann equations have the form \( \frac{df}{d\overline{z}} = 0 \).
Differentiating non-analytic functions

The equation \( df = f_z \, dz + f_{\overline{z}} \, d\overline{z} \)
suggests that \( z \) and \( \overline{z} \) formally
can be considered as independent variables. Indeed, \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \overline{z}} \)
have the same properties as \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \).

Some properties:

1) \( \left( \overline{f} \right)_z = \overline{\left( f_z \right)} \), \( \left( \overline{f} \right)_{\overline{z}} = \overline{\left( f_{\overline{z}} \right)} \)

2) Sum, product, quotient rules hold for \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \overline{z}} \), e.g.,
\[ f \cdot g \right)_z = f_z \cdot g + f \cdot g_z \]

3) Chain rule. Let \( h(z) = f \left( g(z) \right) \), \( w = g(z) \).
\[ h_z = f_w \cdot g_z + f_{\overline{w}} \cdot \left( \overline{g} \right)_z = f_w \cdot g_z + f_{\overline{w}} \cdot \overline{\left( g_z \right)} \]
\[ h_{\overline{z}} = f_w \cdot g_{\overline{z}} + f_{\overline{w}} \cdot \left( \overline{g} \right)_{\overline{z}} = f_w \cdot g_{\overline{z}} + f_{\overline{w}} \cdot \overline{\left( g_{\overline{z}} \right)} \]
Example

\[ f(z) = \sin (z \overline{z}^2) \]

\[ f_z = \cos (z \overline{z}^2) \frac{\partial}{\partial z} (z \overline{z}^2) = \cos (z \overline{z}^2) \overline{z}^2 \]

\[ f_{\overline{z}} = \cos (z \overline{z}^2) \frac{\partial}{\partial \overline{z}} (z \overline{z}^2) = \cos (z \overline{z}^2) z \cdot 2\overline{z} \]

\[ \frac{d}{dz}, \quad \frac{d}{d\overline{z}}, \quad \text{and} \quad \frac{d}{d\overline{z}} \]

The notation \( \frac{d}{dz} \) is used for C-derivative,

\[ f' = \frac{df}{dz} \]

For a smooth function \( f \), \( \frac{df}{dz} \) and \( \frac{df}{d\overline{z}} \) always exist, but \( \frac{df}{dz} = f' \) exists only if \( \frac{df}{d\overline{z}} = 0 \), in which case \( \frac{df}{dz} = \frac{df}{d\overline{z}} \). (Thm above)

Formally, a function is analytic if it does not depend on \( \overline{z} \)!